FRONT PROPAGATION DIRECTED BY
A LINE OF FAST DIFFUSION:
LARGE DIFFUSION AND LARGE TIME ASYMPTOTICS

by Laurent Dietrich & Jean-Michel Roquejoffre

Abstract. — The system under study is a reaction-diffusion equation in a horizontal strip, coupled to a diffusion equation on its upper boundary via an exchange condition of the Robin type. This class of models was introduced by H. Berestycki, L. Rossi and the second author in order to model biological invasions directed by lines of fast diffusion. They proved, in particular, that the speed of invasion was enhanced by a fast diffusion on the line, the spreading velocity being asymptotically proportional to the square root of the fast diffusion coefficient. These results could be reduced, in the logistic case, to explicit algebraic computations. The goal of this paper is to prove that the same phenomenon holds, with a different type of nonlinearity, which precludes explicit computations. We discover a new transition phenomenon, that we explain in detail.

Résumé (Propagation de fronts dirigée par une ligne de diffusion rapide : limite en temps long et grande diffusion)

Nous étudions une équation de réaction-diffusion posée dans une bande horizontale, couplée à une équation de diffusion sur son bord supérieur à travers une condition de Robin. Cette classe de modèles a été proposée par H. Berestycki, L. Rossi et le deuxième auteur afin d'étudier l'influence d'une ligne de diffusion rapide (par exemple une route) sur les invasions biologiques. Ils prouvent que la vitesse d'invasion est augmentée par une forte diffusivité sur la ligne, et plus précisément asymptotiquement proportionnelle à la racine carrée de cette dernière. Dans le cas d'une croissance logistique, ces résultats peuvent être réduits à des calculs algébriques. Le but de cet article est de généraliser ce résultat à des non-linéarités différentes et pour lesquelles ces calculs ne peuvent être accomplis. Nous mettons aussi en lumière un nouveau phénomène de transition entre deux ondes progressives différentes, qu'on explique en détail.

Contents

1. Introduction, statement of the problem

1.1. Model and question. — Let Ω_L be the strip $\mathbb{R} \times (-L, 0)$. The goal of this paper is to study the large time asymptotics of the following system:

$$(1) \quad \begin{cases} u_t - Du_{xx} + \mu u - v(t,x,0) = 0 & (t > 0, \ x \in \mathbb{R}) \\ v_t - d\Delta v = f(v) & (t > 0, \ (x,y) \in \Omega_L) \\ dv_y(t,x,0) + v(t,x,0) = \mu u(t,x) & (t > 0, \ x \in \mathbb{R}) \\ v_y(t,x,-L) = 0 & (t > 0, \ x \in \mathbb{R}). \end{cases}$$

The unknowns are the functions $(u(t,x), v(t,x,y))$, respectively defined on $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+ \times \Omega_L$. The positive numbers μ, d, D are given. The function $f(v)$ is smooth, and there is $\theta > 0$ such that $f \equiv 0$ on $[0, \theta]$ and $f(1) = 0$. Moreover $f > 0$ on $(\theta, 1)$ and $f'(1) < 0$. Such a nonlinear term will sometimes be referred to as *ignition type nonlinearity*, in reference to the mathematical literature on flame propagation models. Of particular interest to us will be the large time asymptotics of (1), combined with the limit $D \to +\infty$.

Remark 1.1. — All of the techniques used in this paper can be used for a *bistable* nonlinearity by replacing $f(v) \geqslant 0$ by $f(v) \geqslant -\mathrm{Lip}(f) \times v$ where needed. Nonetheless, the applicability of the current paper depends on the existence of traveling waves for the above system with speed

$$(2) \quad c(D) \sim_{D \to +\infty} c_\infty \sqrt{D}$$

(see Theorem 2.2). In [11], the first author proves existence of traveling waves for *ignition* and *bistable* nonlinearities, but the proof of the asymptotics (2) found in [12] holds only for *ignition* type. Extending it to bistable type is still open.

The modeling for (1) can be found in [6]. We would like to point out that the boundary condition for v at $y = 0$ is the only one that ensures total mass conservation ($\int_\mathbb{R} u(t,x)\mathrm{d}x + \int_{\Omega_L} v(t,x,y)\mathrm{d}x\mathrm{d}y$) in the absence of nonlinearity ($f = 0$), as can be seen by integrating by parts the equation for v. We turn now to the motivation for and the meaning of (1).

1.2. Motivation. — System (1) was proposed for the first time by Berestycki, Rossi and the second author in [6], as a model for biological invasions in oriented habitats. It was indeed observed in several instances that transportation networks tend to enhance the speed of invasion. Let us mention two biological instances: the pine processionary moves northwards faster than anticipated, and it is believed that the road network has

a responsibility in the phenomenon, see for example [23]. The yellow-legged hornet has invaded the whole South West of France, as is reported in the maps provided in [24]: it first followed the main rivers, and from then colonized the inland areas.

Such boundary couplings between 3D-2D or 2D-1D equations to model transportation networks have been used extensively, see the introduction of [4] for various examples.

In [6], Ω_L is replaced by the whole upper half-plane, and the nonlinearity f is a Fisher-KPP type nonlinearity ($f(0) = f(1) = 0$, $f > 0$ concave between 0 and 1). The line $\{y = 0\}$ is named 'the road', and the upper half-plane is named 'the field'. This terminology will be freely used here. The authors showed the dramatic effect of the road on the overall propagation: there is $c_*(D) > 0$ such that, for all $c < c_*(D)$ we have

$$\lim_{t \to +\infty} \inf_{|x| \leqslant ct} u(t, x) = 1/\mu, \quad \lim_{t \to +\infty} \inf_{|x| \leqslant ct} v(t, x, y) = 1, \quad \text{locally uniformly in } y \in \mathbb{R},$$

and, for all $c > c_*(D)$ we have

$$\lim_{t \to +\infty} \sup_{|x| \geqslant ct} u(t, x) = 0, \quad \lim_{t \to +\infty} \sup_{|x| \geqslant ct} v(t, x, y) = 0, \text{ uniformly in } y \in \mathbb{R}.$$

Moreover, there is $c_\infty > 0$ such that $c_*(D) \sim c_\infty \sqrt{D}$, as $D \to +\infty$. Thus, the road enhances and leads the propagation in its direction, even far away in the field where the species diffuses only with a mobility $d \ll D$. This is in sharp contrast with the classical and homogeneous propagation results for reaction-diffusion equations, such as Aronson-Weinberger [2].

One could question whether this effect is due to the special structure of the Fisher-KPP nonlinearity, or if it holds for more general terms f. Indeed, the Fisher-KPP assumption $f'(0)v \leqslant f(v)$ enables to reduce the question of spreading speed to algebraic computations on the linearized equation near the $(0, 0)$ state, whereas for more general nonlinearities no such computations are available (for an ignition type nonlinearity the linearized would just be the heat equation) and the dynamics have to be studied more carefully. This is the first motivation behind our choice for f. Moreover, f has also an ecological meaning by itself, which is that of a *weak Allee effect* (see [20, §2.2.1.1] for a precise description): broadly speaking, it models species for which the *per capita growth* is positive but lowest for lowest densities.

In [12], the first author gives a first hint of the robustness of the phenomenon discussed above, by constructing traveling waves $(\phi(x + ct), \psi(x + ct, y))$ to (1) with ignition type nonlinearity whose speed c satisfies indeed $c(D) \sim c_\infty \sqrt{D}$, where $c_\infty > 0$ is characterized in terms of a limiting problem obtained by rescaling x by $\sqrt{D}$ and sending D to infinity. In order to confirm the phenomenon of [6] for (1) with the ignition type nonlinearity, one should now understand whether, and how, those traveling waves attract the solutions of (1). This is the aim of this paper. Instead of presenting the results now, we will show some numerical simulations, which reveal a transition phenomenon in the propagation that we had not expected.

1.3. Some numerical simulations. — These simulations were produced using FreeFem++. We used P2 finite elements on a mesh of 400×50 points. The time scheme used is a two-step (to handle the coupling) explicit Euler, which seems quite sufficient in terms of accuracy and speed for our context. Neumann boundary conditions are imposed on the sides of a domain of size $A \times L$ with $A \gg L$. Finally, we represented u as a function over the whole domain so that it is visible. The following parameters were used:

$\mu u_0, v_0$	$\mathbf{1}_{(-3,3)}(x)$
d	0.1
D	100
μ	1.4
θ	0.3
$f(v)$	$10 \times \mathbf{1}_{v>\theta}(v-\theta)^2(1-v)$
A	500
L	50
Δt	0.1

Figure 1. Scale. u, v are solutions of (1). u is extended in a constant way over the all Ω_L for a clearer representation and comparison with v.
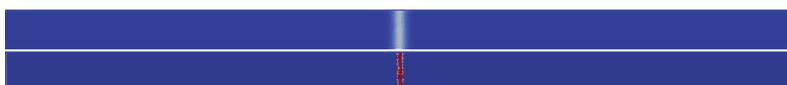
Figure 2. $t = 0$
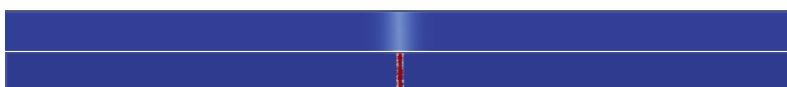
Figure 3. $t = 10\Delta t$
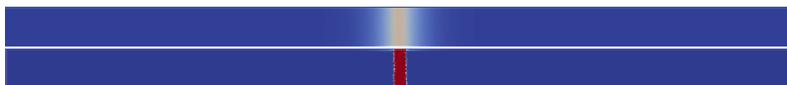
Figure 4. $t = 75\Delta t$
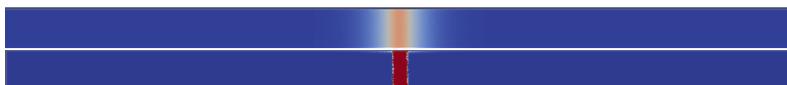
Figure 5. $t = 100\Delta t$
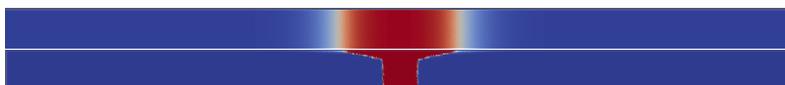
Figure 6. $t = 300\Delta t$

Figure 7. $t = 1000\Delta t$

Figure 8. $t = 1300\Delta t$

The scenario that we would expect is thus the following: due to the large diffusivity D, u is quickly spread on all $\mathbb{R}$ and decays rapidly. Meanwhile, v grows slowly and transmits mass to u. At some point, u has recovered enough mass and starts to lead the propagation. The acceleration of the propagation is then transmitted downwards from the road to the bottom of the field, reaching the regime dictated by the traveling wave. The remainder of this paper is devoted to proving that this is indeed what happens.

2. Main results, discussion

Let us reformulate System (1) in the following way, we hope that it will help the reader visualize the problem.

(3)
$$
\begin{array}{c}
\partial_t u - D\partial_{xx}^2 u = v - \mu u \\
\hline
d\partial_y v = \mu u - v \\[4pt]
\partial_t v - d\Delta v = f(v) \\[4pt]
\partial_y v = 0 \\
\hline
\end{array}
$$

We also want to study the behaviour for large D, so the renormalization $(x \leftarrow x\sqrt{D})$ will often be used:

(4)
$$
\begin{array}{c}
\partial_t u - \partial_{xx}^2 u = v - \mu u \\
\hline
d\partial_y v = \mu u - v \\[4pt]
\partial_t v - \frac{d}{D}\partial_{xx}^2 v - d\partial_{yy}^2 v = f(v) \\[4pt]
\partial_y v = 0 \\
\hline
\end{array}
$$

Some results will be stated for Equation (3) and some for (4) and the proofs will juggle between the two. We would like to emphasize that uniqueness as well as many

properties of this system are a consequence of the monotone structure of (3) inherited from the maximum principle investigated in [6, 11, 12]. The purpose of this paper is to investigate the large-time and large-diffusion asymptotics of this solution. We briefly mention existence and uniqueness of a solution and refer to [6] for the proof (where the strip is replaced by a half-plane but the argument still holds).

THEOREM 2.1 (Stated for Equation (3)). — *Let $(u_0, v_0) \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\Omega_L)$, $0 \leqslant \mu u_0$, $v_0 \leqslant 1$. There exists a global solution in the classical sense to (3) with initial data (u_0, v_0). This solution is unique in the class of bounded classical solutions and satisfies $0 \leqslant \mu u(t, x), v(t, x, y) \leqslant 1$ for all $t \geqslant 0$, $x \in \mathbb{R}$, $y \in \Omega_L$.*

As we will use them throughout the paper, we also mention the results of [11, 12] about traveling waves.

THEOREM 2.2. — *There exists a unique speed $c > 0$ and a unique pair of smooth increasing (in the x variable) profiles $\phi(x), \psi(x, y)$ up to translation in the x-direction, connecting the states $(0, 0)$ and $(1/\mu, 1)$ such that $(\phi(x+ct), \psi(x+ct, y))$ is a traveling wave solution of (3). Moreover, there exists $c_\infty > 0$ such that $c(D) \sim c_\infty \sqrt{D}$ as $D \to +\infty$.*

2.1. FIRST RESULTS. — The following theorems are natural consequences of the stability of front-like initial data, using an argument initiated by [15], and are not so unexpected. They will, nevertheless, be useful for later purposes. A specificity of the present computations is that they should be uniform in the large parameter D, this is why they are detailed.

THEOREM 2.3 (Stated for Equation (3)). — *Let (u_0, v_0) be a front-like initial datum for Equation (4), that is $(u_0, v_0) \in \mathcal{P}_{\alpha_0}$ defined in the next section. There exists an exponent $\omega > 0$ that depends on the initial data only through α_0, and for all $\varepsilon > 0$ small enough there exist two shifts $\xi_1^{\pm} \in \mathbb{R}$ such that*

$$\mu\phi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} \leqslant \mu u(t, x) \leqslant \mu\phi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t}$$
$$\psi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} \leqslant v(t, x, y) \leqslant \psi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t},$$

where C is a constant that depends only on f, d and L. Moreover, ω does not depend on $D > d$.

REMARK 2.1. — The previous theorem does not give the convergence towards traveling waves, but it gives a precise spreading velocity. In [21], this is the starting point of an iterative argument showing a geometric decrease of the distance separating the two shifts with respect to a fixed time step. One could think of adapting the argument of [21] to (4), but this would not be uniform in $D > d$. We prefer to focus on the really new features of the model.

We now turn to what happens for compactly supported initial data.

Theorem 2.4 (Stated for Equation (3)). — *Let (u_0, v_0) be non-negative smooth compactly supported data. There exist $\delta > 0$ and $M = O(\sqrt{D})$ such that if $\mu u_0, v_0 > 1 - \delta$ for $(x, y) \in (-M, M) \times [-L, 0]$ then $\mu u, v$ stays trapped (in the sense of Theorem 4.1) up to an exponentially decaying error between two shifts of a pair of traveling waves evolving in both directions.*

2.2. Data with $O(1)$ support and additional effects. — Throughout the rest of the paper, *extinction* will mean convergence of the solution towards $(0, 0)$ locally uniformly in space as $t \to +\infty$ and *invasion* or *propagation*, convergence of the solution towards $(1/\mu, 1)$ locally uniformly in space as $t \to +\infty$. This vocabulary is inspired by the ecological interpretation of the equations. Of special interest to us will be the description of *how* invasion happens when it does.

In this section we state the results that account for the above numerical simulations. This first result says that one only needs v_0 to be close enough to 1 on a rectangle $(-M', M') \times [-L, 0]$ independent of D to initiate propagation, and after a waiting time t_D one recovers fast propagation to the traveling wave speed as in Theorem 2.4.

Theorem 2.5 (Stated for Equation (3)). — *Let L be large enough (independently of D). There exist $M', \delta' > 0$ independent of $D > d$ such that, if the initial datum $0 \leqslant \mu u_0, v_0 \leqslant 1$ satisfies*

$$v_0 > 1 - \delta' \quad \text{for } x \in (-M', M'),$$

then the following holds: let $h(D)$ be any infinitely increasing function as $D \to \infty$; then after a time $t_D = D^{1/2} h(D) + O(1)$, the functions μu and v satisfy the assumptions of Theorem 2.4, in other words:

$$\mu u, v \geqslant 1 - \delta \quad \text{for } x \in (-M\sqrt{D}, M\sqrt{D}).$$

As a consequence, starting from the time $t = t_D$, propagation occurs as described in Theorem 2.4. On the numerical simulations, this moment would be around $1000\Delta t$, see Figure 7, when the level lines of v are completely inclined. The solution actually starts to accelerate before that – see Figure 6, around $300\Delta t$ – but this acceleration takes also some time to propagate from $y = 0$ to $y = -L$. The waiting time t_D accounts for both the first slow propagation phase with flat level lines plus the time needed to fully transmit the acceleration from the road.

One could argue that this happens in a much smaller time. The next theorem shows that, even if the solution may not take all the time t_D to fall into the assumptions of Theorem 2.4, it stills needs a lot of time. To what extent the upper bound in the above theorem, and the lower bound in the next theorem, can be reconciled, is a very interesting question that we leave for future work.

Theorem 2.6 (Stated for Equation (3)). — *Let $M', \delta' > 0$ be as in Theorem 2.5. For every $0 < \kappa < 1/7$ small, there exists $C_\kappa > 0$ such that, if*

$$(5) \qquad\qquad t \leqslant C_\kappa D^{1/7 - \kappa},$$

then $(\mu u(t,.), v(t,.,.))$ does not satisfy the assumptions of Theorem 2.4. More precisely we have, uniformly in t satisfying (5):

$$\lim_{D \to +\infty} \|u(t,.)\|_\infty = 0, \qquad \lim_{D \to +\infty} \frac{\left|\{(x,y) \in \Omega_L : \ v(t,x,y) \geqslant \theta\}\right|}{\sqrt{D}} = 0.$$

Finally, we investigate the situation of an initial datum supported only on the road. The behaviour that we find does not at all look like what we have just discovered for initial data supported in the field. If $\mu u_0 \leqslant 1$ has a support of size $\leqslant C\sqrt{D}$ there will be extinction. On the other hand, we also provide conditions on μ for invasion to happen.

THEOREM 2.7 (Stated for Equation (4)). — *Let $v_0 \equiv 0$ and $\mu u_0 = \mathbf{1}_{(-a,a)}$ be initial data for* (4) *and u, v the associated solutions. We have the following :*

   – There exists $a_0 > 0$ independent of D such that if $a < a_0$, μu and v decay to 0 uniformly as $t \to +\infty$.

   – If $a = +\infty$ there are thresholds $\mu^\pm$ independent of D such that for $\mu < \mu^-$ invasion occurs and for $\mu > \mu^+$, μu and v converge uniformly to $1/(\mu(L + 1/\mu)) \leqslant \theta$.

   – More generally, provided $\mu < \mu^-$, there exists $a_1 > 0$ independent of D such that if $a > a_1$, invasion occurs.

REMARK 2.2. — It is quite natural that μ too large leads to extinction: indeed, we normalized u so that $u \leqslant 1/\mu$ and moreover μ acts as a death rate in the equation on u. Meanwhile, v sees the same initial boundary Robin condition $\equiv 1$ independently of μ.

The case $a = +\infty$ (fully crowded road) does not have much biological meaning by itself, but we use it as a tool to study the case of large a, as the associated solutions are close to each other in exponentially weighted spaces.

2.3. BIBLIOGRAPHICAL STUDY AND DISCUSSION. — The general issue of our work is that of speed-up versus quenching. The first contribution concerning the behaviour of compactly supported initial data in reaction-diffusion equation of ignition (or bistable) type can be found in Kanel' [18]. For the one dimensional equation

$$\partial_t v - \partial_{xx}^2 v = f(v)$$

the author shows the existence of two thresholds $0 < L_0 \leqslant L_1 < +\infty$ such that if $v_0 = \mathbf{1}_{(-l,l)}$ with $l < L_0$, v ends up below θ in finite time (and as a consequence, decays to 0 uniformly) : we call this situation *quenching*. On the other hand, if $l > L_1$ it is shown that $v \to_{t \to +\infty} 1$ uniformly on compact sets. Zlatoš [26] showed $L_0 = L_1$, and more generally Du and Matano [13] showed the existence of such thresholds for general one-parameter families of initial data.

For equations in cylinders and in the presence of a parallel shear flow,

$$\partial_t v + A\alpha(y)\partial_x v - \Delta v = f(v)$$

an important issue is to understand how a large amplitude flow (i.e., $A \gg 1$) will enhance spreading. This has been studied in various papers starting from [3], where

it is shown in the case of a Fisher-KPP nonlinearity, a linear speed-up

$$c_*(A) \underset{A \to +\infty}{\sim} kA.$$

In a more general setting, let us quote Constantin-Kiselev-Oberman-Ryzhik [7], who introduce the notion of *bulk burning rate*. For ignition type nonlinearities, the same result holds as proved by Hamel and Zlatoš [17] (see [12] for a comparison of their result with our situation). As for whether propagation or quenching holds, Constantin-Kiselev-Ryzhik [8] and Kiselev-Zlatoš [19] show that the price to pay for propagation (hence, speed-up) also is a linear scaling in A for the initial size of temperature distribution:

$$L_0 \underset{A \to +\infty}{\sim} k_0 A, \quad L_1 \underset{A \to +\infty}{\sim} k_1 A,$$

provided that the flow is not constant on too large intervals. In other words, one trades a linear (in the amplitude of the flow) speed up of propagation for a similar linear growth in the critical size of initial data that leads to quenching.

In the case of cellular flows, the same phenomenon happens but with a scaling in $A^{1/4}$ (up to a logarithmic factor): the speed-up property was proved by Novikov and Ryzhik [22] for the KPP case and more recently by Zlatoš [27] for combustion type nonlinearities. On the other hand, Fannjiang-Kiselev-Ryzhik [14] proved (for flows with small enough cells) that if $L^4 \ln(L) < kA$ – where L represents the size of the square supporting the initial datum – quenching happens. See also the numerical simulations of [25].

A different type of mechanism is studied in Constantin-Roquejoffre-Ryzhik-Vladimora [9] where the authors investigate a system coupling a reaction-diffusion equation and a Burgers equation. They show different quenching results with respect to a gravity parameter, one of them being that quenching happens independently of l when the gravity is large enough.

In the light of this section, Theorem 2.5 may come up as a surprise since it shows a speed-up of the propagation (to an asymptotic speed $c = c_\infty \sqrt{D}$) for free: D does not appear in the threshold size of the initial data v_0 whereas in the above described context, the initial distribution of temperature really needs to be above a plateau large enough whose size scales as the amplitude of the flow to some power. The trade-off in our setting is the presence of a "two-speed" mechanism: propagation first happens at a small speed that does not depend on D, but accelerates towards the full speed $c(D)$. On the other hand, if one tries to initiate the invasion only thanks to $\mu u_0 = \mathbf{1}_{(-l,l)}$ and $v_0 \equiv 0$, Theorem 2.7 shows that quenching happens if $l < a_0 D^{1/2}$ (from the point of view of (3)).

2.4. **Organization of the paper.** — Section 3 is devoted to proving Theorem 2.3 in the more precise form Theorem 3.1, Section 4 provides the details for the proof of Theorem 2.4 by proving the detailed Theorem 4.1. In section 5 we prove Theorem 2.5, and we prove Theorem 2.6 in Section 6. These two sections will describe more precisely the mechanism that is at work. Finally, the last section investigates the case of initial data supported on the road only.

3. Front-like initial data

Let us first trap the initial data between functions that will evolve in sub and super-solutions traveling at the right speed.

3.1. Trapping the initial data. — We deal with bounded, uniformly continuous perturbations (ρ_1, ρ_2) such that there exist C and $\alpha_0 > 0$ such that $\rho_i(x) \leqslant Ce^{\alpha_0 x}$. Then we assume

(6) $0 \leqslant \mu u_0, v_0 \leqslant 1$

(7) $(u_0, v_0) = (\phi(x+\xi), \psi(x+\xi)) + (\rho_1, \rho_2)$

for some (ρ_1, ρ_2) of the above form and some translation $\xi \in \mathbb{R}$. Such initial data is said to be in the class $\mathcal{P}_{\alpha_0}$. In this subsection, we prove that such initial data can be trapped between two translates of the traveling front, which is conceptually simple but necessary. Due to the degeneracy of $f(v)$ as $v \leqslant \theta$, we will have to use the following weight function. Let $L_0 > 3$ and

$$0 < \alpha < \min(\alpha_0, c).$$

Define $\Gamma(x)$ to be a smooth non-decreasing function such that

(8) $\Gamma(x) = \begin{cases} 1 & \text{if } x > L_0 \\ e^{\alpha(x+L_0)} & \text{if } x < -L_0 - 1. \end{cases}$

We also recall the exponential convergence towards 0 or 1 as $x \to \pm\infty$ proved in [11], [12]: there exist $\lambda, \widetilde{\lambda} > 0$ (bounded from below uniformly in $D > d$) and one can enlarge $L_0 > 0$ so that

(9)
$$\forall x < -L_0/2 \qquad \mu\phi, \psi \leqslant \quad \frac{\theta}{2} e^{\lambda(x+L_0/2)} \quad \leqslant \frac{\theta}{2}$$
$$\forall x > L_0/2 \qquad 1 - \mu\phi, 1 - \psi \leqslant \frac{1-\theta_1}{2} e^{-\widetilde{\lambda}(x-L_0/2)} \leqslant \frac{1-\theta_1}{2},$$

where $\theta < \theta_1 < 1$ is chosen so that $-f'(s) \geqslant -f'(1)/2 =: \beta > 0$ when $s > \theta_1$. That way, ahead of the front the system becomes linear and behind the front one controls the monotonicity of f. We now assert the following.

Proposition 3.1. — *Assume* (6), (7). *Then for any* $\varepsilon > 0$, *there exist* $\xi_0^- < 0$ *and* $\xi_0^+ > 0$ *large enough such that*

(10) $\mu\phi(x+\xi_0^-) - \varepsilon\Gamma(x+\xi_0^-) \leqslant \mu u_0(x) \leqslant \mu\phi(x+\xi_0^+) + \varepsilon\Gamma(x+\xi_0^+)$

(11) $\psi(x+\xi_0^-, y) - \varepsilon\Gamma(x+\xi_0^-) \leqslant v_0(x,y) \leqslant \psi(x+\xi_0^+, y) + \varepsilon\Gamma(x+\xi_0^+).$

Proof. — We only prove (10). (11) is obtained simultaneously with the same arguments (y-uniform limits, y-uniform exponential decay) by taking $|\xi_0^\pm|$ large enough. We start with the right inequality. Let $\varepsilon > 0$. Thanks to the uniform limit of ϕ as $x \to +\infty$, there exists B_ε independent of ξ_0^+ such that for $x \geqslant -\xi_0^+ + L_0 + B_\varepsilon$,

$$\mu\phi(x+\xi_0^+) + \varepsilon\Gamma(x+\xi_0^+) \geqslant 1 - \varepsilon + \varepsilon = 1 \geqslant \mu u_0(x).$$

On the other hand, when $x \leqslant -\xi_0^+ - L_0 - 1$, $\mu u_0(x) \leqslant Ce^{\alpha_0 x}$, so here for the inequality to be true one needs $\varepsilon e^{\alpha(x+\xi_0^+ +L_0)} \geqslant Ce^{\alpha_0 x}$. But since $x+\xi_0^+ +L_0 < 0$ and $0 < \alpha < \alpha_0$, one just needs $\varepsilon e^{\alpha_0(x+\xi_0^+ +L_0)} \geqslant Ce^{\alpha_0 x}$, which is ensured as soon as $\xi_0^+ > \ln(C/\varepsilon) - L_0$.

Now only the compact region $x \in [-\xi_0^+ - L_0 - 1, -\xi_0^+ + L_0 + B_\varepsilon]$ remains. Observe that on this interval, $\mu u_0(x)$ goes uniformly to 0 as $\xi_0^+ \to \infty$, whereas the right-hand side in (10) has a fixed positive infimum, so that the desired order is obtained by enlarging ξ_0^+.

For the existence of ξ_0^-: observe that on $x \geqslant -\xi_0^- + L_0 + 1$

$$\mu \phi(x + \xi_0^-) - \varepsilon \Gamma(x + \xi_0^-) \leqslant 1 - \varepsilon \leqslant \mu u_0(x),$$

provided ξ_0^- is negative enough, thanks to the uniform limit of u_0 as $x \to +\infty$. Now for the rest of the proof, we need on $x \leqslant -\xi_0^- + L_0 + 1$

$$\varepsilon e^{\alpha(x+\xi_0^- +L_0)} \geqslant \mu \phi(x + \xi_0^-) - (\mu \phi(x) + \rho(x)).$$

Because the exponential decay λ of ϕ and ψ satisfies $\lambda > c \geqslant \alpha$ (see [12]) this is true on $x \leqslant -\xi_0^- - L_0 - B_\varepsilon$ with $B_\varepsilon > 0$ large enough independent of ξ_0^- so that here

$$\varepsilon e^{\alpha(x+\xi_0^- +L_0)} \geqslant \theta e^{\lambda(x+\xi_0^-)} \geqslant \mu \phi(x + \xi_0^-).$$

Again, we cover the compact region left around the interface by enlarging $-\xi_0^-$. □

3.2. WAVE-LIKE SUB AND SUPERSOLUTION. — We adapt the original result of Fife-McLeod [15] using the simplified notation and generalization of Mellet-Nolen-Ryzhik-Roquejoffre [21]. The adaptation is computationally non trivial, so let us first explain the changes that we expect to happen. Our objective is to build a supersolution $\overline{u}, \overline{v}$ to (4) that is close to the front (ϕ, ψ) (in the frame moving at speed c). In the homogeneous case and for generalized transitions fronts the authors of [21] proposed $\overline{v} = \psi(x + c\xi(t)) + q(t)\Gamma(x + c\xi(t))$, where ψ is the front, $\xi(t)$ is an increasing shift starting from the initial one and converging to some finite limit, $q(t) = \varepsilon e^{-\omega t}$ and Γ is defined above: this is a necessary correction to take into account the initial perturbation and the degeneracy of f on $v \leqslant \theta$. In our case, we will look for

$$(12) \qquad \begin{cases} \mu \overline{u} = \mu \phi(x + c\xi(t)) + q_u(t)\Gamma(x + c\xi(t)) \\ \overline{v} = \psi(x + c\xi(t), y) + q_v(t, y)\Gamma(x + c\xi(t)) \end{cases}$$

and

$$(13) \qquad \begin{cases} \mu \underline{u} = \mu \phi(x - c\xi(t)) - q_u(t)\Gamma(x - c\xi(t)) \\ \underline{v} = \psi(x - c\xi(t), y) - q_v(t, y)\Gamma(x - c\xi(t)) \end{cases}$$

with ξ starting from $\xi_0^\pm$. We will also use the fact that

$$\forall M > 0, \exists \delta_M > 0 \mid \partial_x \phi, \partial_x \psi > \delta_M \text{ when } x \in (-M, M).$$

Now we reduce α a bit more and set $\alpha = \min(\alpha_0, c/5)$, just so that the quantities

$$\alpha c - d/D\alpha^2 \geqslant \alpha c - \alpha^2 > \alpha c/4 - \alpha^2 > 0$$

cannot be zero. These quantities will play an important role in the following computa-
tions. Observe that this condition on α means that the decay correction obtained
through Γ is limited: solutions starting from large perturbations (i.e., small α_0) will
be stabilized thanks to a correction with an α_0 decay also, but solutions from very
small perturbations (i.e., very large α_0) will still need a $c/5$ correction in the decay
at $-\infty$ to be stabilized.

Since we still want an exponential decay of $q_v(t)$ we look for

$$(14) \qquad \begin{cases} q_u(t) = \varepsilon C e^{-\omega t} \\ q_v(t,y) = \varepsilon h(y) e^{-\omega t} \end{cases}$$

with separate variables. The boundary conditions yield $h'(-L)=0$ and $h'(0)+h(0)=C$
so that we have a large choice for h. Nonetheless, it will become clear in the following
computations that a good candidate is

$$(15) \qquad C = \cosh\left(\sqrt{\kappa/d}\,L\right) + \sinh\left(\sqrt{\kappa/d}\,L\right), \quad h(y) = \cosh\left(\sqrt{\kappa/d}\,(y+L)\right)$$

with

$$\kappa = \min\left(\beta/2, (\alpha c - d/D\alpha^2)/2\right) > 0, \quad \omega = \min\left(G(\sqrt{\kappa/d}\,L), \beta/2, \mathrm{Lip}\,f, \alpha c/4 - \alpha^2\right) > 0$$

and $G(x) = \mu \tanh(x)/(1 + \tanh(x))$. The role of these conditions will be clear in the
computations.

Observe that since $G'(0) > 0$, the decay exponent ω is then linearly small as β
or α_0 or μ is small, but it should be noticed that it does not depend on $D \geqslant d$, and
that it depends on the initial data only through α_0. We can now state the following.

Theorem 3.1. — *Assume* (6), (7) *and let u, v denote the associated solutions of* (4).
Let $\varepsilon_0 = \min(\theta/4, (1-\theta_1)/4, \gamma_0)$ where

$$\gamma_0 = \frac{1}{4B}, \quad B = \left(\frac{3\,\mathrm{Lip}\,f + |\Gamma|_{C^2}}{c\delta_{L_0+2}}\right) C \max(1, 1/\mu).$$

*There exists a constant K_0 that depends on the initial data only through α_0 and such
that if $\varepsilon \in (0, \varepsilon_0)$, there exists $\xi_1^{\pm}$ with*

$$(16) \qquad \xi_1^+ \leqslant \xi_0^+ + \varepsilon K_0, \quad \xi_1^- \geqslant \xi_0^- - \varepsilon K_0$$

and for all $t \geqslant 0$,

$$(17) \qquad \phi(x + c\xi_1^-) - q_u(t)\Gamma(x + c\xi_1^-) \leqslant \mu u(t, x - ct)$$
$$\leqslant \mu \phi(x + c\xi_1^+) + q_u(t)\Gamma(x + c\xi_1^+)$$

$$(18) \qquad \psi(x + c\xi_1^-) - q_v(t,y)\Gamma(x + c\xi_1^-) \leqslant v(t, x - ct, y)$$
$$\leqslant \psi(x + c\xi_1^+) + q_v(t,y)\Gamma(x + c\xi_1^+).$$

Proof. — Inequations (17), (18) are stated in the moving frame with variables
$(t, x + ct)$. As a consequence, in the computations one has to replace ∂_t by $\partial_t + c\partial_x$.
We now want to show that $\overline{u}, \overline{v}$ as defined in (12) yields indeed a supersolution:

$$\mathcal{N}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
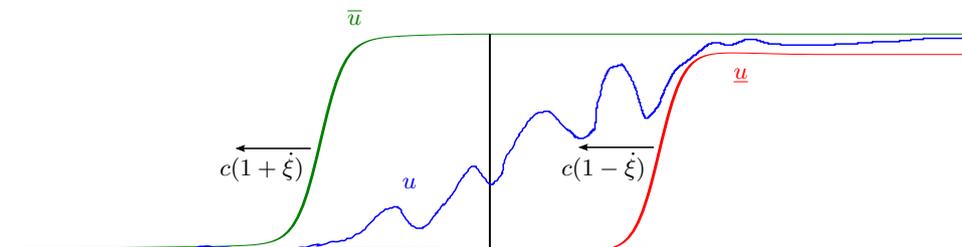
Figure 9. Trapping of the front-like data

where
$$\mathcal{N}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t - u_{xx} + cu_x + \mu u - v(\cdot,0) \\ v_t - \frac{d}{D}v_{xx} - dv_{yy} + cv_x - f(v) \end{pmatrix}$$

and that $\underline{u}, \underline{v}$ as defined in (13) yields a subsolution. Then (17), (18) will follow by an application of the comparison principle, Proposition 3.1, and the monotonicity of ξ. Indeed, this will show that in the original frame u, v stays trapped between the fronts shifted initially by $\xi_0^\pm$ and moving at speed resp. $c(1\pm\dot\xi)$ (or speed $c\pm\dot\xi$ in the moving frame). This deformation becomes of course exponentially small over time due to the $e^{-\omega t}$ factor. Observe also that $\dot\xi \leqslant 1/4$ and is exponentially decaying over time, so u, v will propagate at least and at most with speed $c + o(1)$.

We divide this computation in three zones concerning $x + \xi(t)$. In the following, ϕ and ψ will always mean $\phi(x + c\xi(t))$ and $\psi(x + c\xi(t), y)$, q_u will always mean $q_u(t)$, q_v will mean either $q_v(t,0)$ or $q_v(t,y)$ and Γ will always mean $\Gamma(x+c\xi(t))$, all of these functions being defined as above in (8) and (14)-(15).

3.2.1. *Behind the front:* $x + \xi(t) > L_0 + 1$. — Here $\Gamma \equiv 1$ and $\psi, \overline{v} \geqslant (1+\theta_1)/2$, so that

$$\begin{aligned}
\mathcal{N}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_1 &= c\dot\xi\phi_x + \dot q_u/\mu - \phi_{xx} + c\phi_x + \mu\phi + q_u - \psi - q_v(t,0) \\
&= c\dot\xi\phi_x + \dot q_u/\mu + q_u - q_v \\
&\geqslant \dot q_u/\mu + q_u - q_v \\
&= \varepsilon e^{-\omega t}\big(-Cw/\mu + C - \cosh(\sqrt{\kappa/d}\,L)\big) \\
&= \varepsilon e^{-\omega t}\big(-\big(\cosh(\sqrt{\kappa/d}\,L) + \sinh(\sqrt{\kappa/d}\,L)\big)w/\mu + \sinh(\sqrt{\kappa/d}\,L)\big) \geqslant 0
\end{aligned}$$

The first inequality holds because we look for $\dot\xi \geqslant 0$ and the last because $\omega \leqslant G(\sqrt{\kappa}\,L)$.

$$\begin{aligned}
\mathcal{N}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_2 &= c\dot\xi\psi_x + \dot q_v - d/D\psi_{xx} + c\psi_x - d\psi_{yy} - d\partial_{yy}^2 q_v - f(\psi) + f(\psi) - f(\overline{v}) \\
&= c\dot\xi\psi_x + \dot q_v + f(\psi) - f(\overline{v}) - d\partial_{yy}^2 q_v \\
&\geqslant \dot q_v - d\partial_{yy}^2 q_v + \beta q_v = \varepsilon e^{-\omega t}h(y)(-\omega - \kappa + \beta) \\
&\geqslant \varepsilon e^{-\omega t}h(y)(-\omega + \beta/2) \geqslant 0.
\end{aligned}$$

The last inequality holds because $w \leqslant \beta/2$ and the next to last because $\kappa \leqslant \beta/2$.

3.2.2. *Ahead of the front: $x + \xi(t) < -L_0 - 1$.* — Heres $\Gamma(x + \xi(t)) = e^{\alpha(x+\xi(t)+L_0)}$, $\psi \leqslant \theta/2$ and $\overline{v} \leqslant \psi + \varepsilon \leqslant 3\theta/4 \leqslant \theta$, so $f(\overline{v} \equiv 0$.

$$\mathcal{N}\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}_1 = c\dot{\xi}\phi_x + \left(\frac{\dot{q}_u}{\mu} + \frac{q_u}{\mu}\alpha c\dot{\xi} - \frac{q_u}{\mu}\alpha^2 + c\frac{q_u}{\mu}\alpha + q_u - q_v(\cdot,0)\right)e^{\alpha(x+\xi+L_0)}$$
$$\geqslant \frac{1}{\mu}(\dot{q}_u + q_u\alpha c\dot{\xi} - q_u\alpha^2 + c\alpha q_u)e^{\alpha(x+\xi+L_0)}$$
$$\geqslant \frac{1}{\mu}\left(-\omega + \alpha c - \alpha^2\right)e^{\alpha(x+\xi+L_0)}q_u \geqslant 0.$$

The last inequality holds because $\omega \leqslant (\alpha c - \alpha^2)/2$, and the first because $q_u(t) \geqslant q_v(t,0)$.

$$\mathcal{N}\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}_2 = c\dot{\xi}\psi_x + e^{\alpha(x+\xi+L_0)}\left(\dot{q}_v + q_v(\alpha(c\dot{\xi}+c) - d/D\alpha^2) - d\partial_{yy}^2 q_v\right)$$
$$\geqslant e^{\alpha(x+\xi+L_0)}q_v(-\omega + \alpha c\dot{\xi} + \alpha c - d/D\alpha^2 - \kappa)$$
$$\geqslant e^{\alpha(x+\xi+L_0)}q_v(-\omega + (\alpha c - d/D\alpha^2)/2) \geqslant 0.$$

The last inequality holds because of the condition on ω, and the next to last because of the condition on κ and because $\dot{\xi} \geqslant 0$.

3.2.3. *The middle region: $|x + \xi(t)| < L_0 + 2$.* — We have

$$\mathcal{N}\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}_1 = c\dot{\xi}\phi_x + \frac{\dot{q}_u}{\mu}\Gamma + c\dot{\xi}\frac{q_u}{\mu}\Gamma_x - \frac{q_u}{\mu}\Gamma_{xx} + c\frac{q_u}{\mu}\Gamma_x + (q_u - q_v)\Gamma$$
$$\geqslant c\dot{\xi}\phi_x + \frac{\dot{q}_u}{\mu}\Gamma - \frac{q_u}{\mu}\Gamma_{xx} \geqslant c\dot{\xi}\delta_{L_0+2} - (\omega + |\Gamma|_{C^2})\frac{q_u}{\mu} \geqslant 0,$$

provided

$$(19) \qquad \dot{\xi} \geqslant \frac{\omega + |\Gamma|_{C^2}}{c\delta_{L_0+2}}q_u.$$

We also have

$$\mathcal{N}\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}_2 \geqslant c\dot{\xi}\psi_x - q_v\operatorname{Lip} f + \dot{q}_v\Gamma + c\dot{\xi}q_v\Gamma_x - d/Dq_v\Gamma_{xx} + cq_v\Gamma_x - d\partial_{yy}^2 q_v\Gamma$$
$$\geqslant c\dot{\xi}\delta_{L_0+2} - q_v\operatorname{Lip} f - \omega q_v - d/D|\Gamma|_{C^2}q_v - \kappa q_v \geqslant 0,$$

provided

$$(20) \qquad \dot{\xi} \geqslant \frac{\operatorname{Lip} f + \omega + d/D|\Gamma|_{C^2} + \kappa}{c\delta_{L_0+2}}q_v.$$

We obtain conditions (19), (20) by remarking that $\kappa \leqslant \beta < \operatorname{Lip}(f)$, $\omega < \operatorname{Lip}(f)$ and $d/D < 1$ and then we take $\dot{\xi}(t) = B\varepsilon e^{-\omega t}$, so that

$$\xi(t) = \xi_0^+ + \frac{B\varepsilon(1 - e^{-\omega t})}{\omega}, \quad K_0 = B/\omega$$

answer our queries. One should observe that the condition $B\varepsilon \leqslant 1/4$ has not been used yet as well as $\omega \leqslant c\alpha/4 - \alpha^2$ rather than just $1/2(c\alpha - \alpha^2)$. Observe that the computations concerning the subsolution (13) with this time

$$\xi(t) = -\xi_0^- + \frac{B\varepsilon(1 - e^{-\omega t})}{\omega}$$

are exactly symmetric, except for a $c\alpha(1 - \dot{\xi})$ term (instead of $c\alpha(1 + \dot{\xi})$) that appears ahead of the front and in the middle region, which is treated thanks to the above still unused assumptions:

$$
\begin{aligned}
-\omega + c\alpha(1 - \dot{\xi}) - d/D\alpha^2 - \kappa &\geqslant -\omega + \frac{3c\alpha}{4} - d/D\alpha^2 - \kappa \\
&\geqslant -\omega + \frac{3c\alpha}{4} - \alpha^2 - \kappa \\
&\geqslant -\omega + \frac{c\alpha}{4} - \frac{\alpha^2}{2} \\
&\geqslant -\omega + \frac{c\alpha}{4} - \alpha^2 \geqslant 0.
\end{aligned}
$$

On the other hand, we have

$$-\omega + c\alpha(1 - \dot{\xi}) - \alpha^2 \geqslant -\omega + \frac{3c\alpha}{4} - \alpha^2 \geqslant -\omega + \frac{c\alpha}{4} - \alpha^2 \geqslant 0.$$

This ends the construction. □

4. Compactly supported initial data

In this section, we go back to the fixed original frame. Seeing the problem in the light of [15] it is natural to test:

$$\begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} \phi(x + ct + c\xi(t)) + \phi(-x + ct + c\xi(t)) - 1/\mu \\ \psi(x + ct + c\xi(t)) + \psi(-x + ct + c\xi(t)) - 1 \end{pmatrix}$$

as a subsolution to (4), i.e., a pair of waves evolving in opposite directions. Of course, in light of the previous section, for this to be a subsolution one needs a well chosen correction in time and in space (in the degeneracy regime of f). Let us define the symmetrized fronts

$$\widetilde{\phi}(\cdot) = \phi(-\cdot), \quad \widetilde{\psi}(\cdot) = \psi(-\cdot).$$

In the sequel we will always use the following notation:

$$\phi = \phi(x + ct + \xi_0 - c\xi(t)), \quad \widetilde{\phi} = \widetilde{\phi}(x - ct - \xi_0 + c\xi(t)),$$

and the same will hold for $\psi, \widetilde{\psi}, \Gamma, \widetilde{\Gamma}$. Here ξ_0 will be a large initial shift and $\xi(t)$ a time-increasing shift with $\xi(0) = 0$ and $c\xi(+\infty) \leqslant 1$, which will be realized as a smallness condition on ε_0. In this section we set

$$(21) \qquad\qquad\qquad \alpha = \min(\lambda, \widetilde{\lambda}, c/5),$$

where λ and $\widetilde{\lambda}$ are already defined in (9) so that α yields the same inequations as above and moreover $\alpha < \lambda, \widetilde{\lambda}$. Γ is defined as above, only with a little more margin. Precisely, let us set this time:

$$(22) \qquad \Gamma(x) = \begin{cases} 1 & \text{if } x > L_0 - 1, \\ e^{\alpha(x+L_0)} & \text{if } x < -L_0 + 1. \end{cases}$$

We will set the following:

$$\underline{u} = \max\big(0, \phi + \widetilde{\phi} - 1/\mu - q_u(t)/\mu \min(\Gamma, \widetilde{\Gamma})\big),$$
$$\underline{v} = \max\big(0, \psi + \widetilde{\psi} - 1 - q_v(t,y) \min(\Gamma, \widetilde{\Gamma})\big).$$

The proof will consist in adapting the previous computations. We shall see that $(\underline{u}, \underline{v})$ yields a subsolution provided only a size condition on the initial shift ξ_0 (independently of $D > d$). This condition is important, because then for the initial data to lie above $(\underline{u}(0), \underline{v}(0))$ it has to be large enough on a large enough interval. Moreover, we would like to insist on the fact that to retrieve the original model (3) one has to change the variable $x \leftarrow x/\sqrt{D}$. As a consequence, when stated for (3), our result assumes that u_0, v_0 are large enough on an interval with length of order $\sqrt{D}$. Theorem 2.4 will be proved as soon as we have proved the

Theorem 4.1

(1) *There exist $\varepsilon_0 > 0$ small enough and two constants $B, \xi_0 > 0$ large enough such that for all $0 < \varepsilon < \varepsilon_0$, there exist a small $\delta > 0$ and $M > 0$ such that if $0 \leqslant \mu u_0, v_0 \leqslant 1$ satisfy $\mu u_0, v_0 > 1 - \delta$ on $x \in (-M, M)$, then*

$$\underline{u} = \max(0, \phi + \widetilde{\phi} - 1/\mu - q_u/\mu \min(\Gamma, \widetilde{\Gamma})), \quad \underline{v} = \max(0, \psi + \widetilde{\psi} - 1 - q_v \min(\Gamma, \widetilde{\Gamma}))$$

where $q_u = \varepsilon C e^{-\omega t}$ and $q_v = \varepsilon h(y) e^{-\omega t}$ are defined as above and this time

$$\xi(t) = \frac{B\varepsilon(1 - e^{-\omega t})}{\omega}$$

defines a subsolution to (4) with initial data u_0, v_0 for all times. By the comparison principle, we then have at all times

$$\underline{u} \leqslant u, \underline{v} \leqslant v.$$

As a consequence, (4) propagates the initial data u_0, v_0 along the x-axis with speed at least as $c + \underset{t \to +\infty}{o}(1)$ in both directions.

(2) *Using the notation of Section 2, we have the following: let $\widetilde{u}, \widetilde{v}$ denote the same functions as in (12) with ϕ, ψ and Γ replaced by $\widetilde{\phi}, \widetilde{\psi}, \widetilde{\Gamma}$. As a consequence, $\widetilde{u}, \widetilde{v}$ will be a supersolution for decreasing front-like initial data. Up to enlarging the initial shifts, we assert that*

$$(\min(\overline{u}, \widetilde{u}), \min(\overline{v}, \widetilde{v}))$$

is a supersolution to (4) with initial data u_0, v_0 for all times. Again, this implies that

$$u \leqslant \min(\overline{u}, \widetilde{u}), v \leqslant \min(\overline{v}, \widetilde{v})$$

and so that the level lines of u, v propagate at most as $c + \dot{\xi} = c + o(1)$ in both directions along the x-axis.

Remark 4.1

(i) As noticed above, observe that one needs to replace $M \leftarrow M\sqrt{D}$ when Theorem 4.1 is stated for the original system (3).

(ii) The size condition on u_0, v_0 is far from optimal and ensures only that $u_0 \geqslant \underline{u}(0), v_0 \geqslant \underline{v}(0)$. It could be sharpened by replacing $1 - \delta$ with θ and by waiting long enough for the reaction to put u, v above $1 - \delta$.
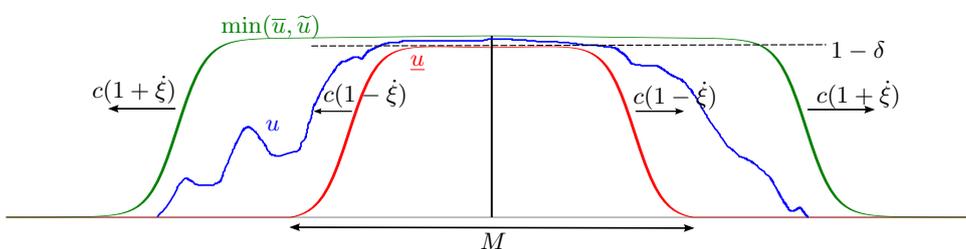
Figure 10. Trapping of the compactly supported data

Proof. — The second part of Theorem 4.1 is easy because the minimum of two supersolutions is a supersolution and any front like initial data can be translated above any compactly supported initial data.

The first part is more intricate. Observe that $\underline{u}(0), \underline{v}(0)$ are zero except on a set of length $(-M, M)$ (with M proportional to ξ_0) and that on $(-M, M)$ they are less than some $1 - \delta$: this directly gives the largeness condition asked so that $u_0 \geqslant \underline{u}(0)$, $v_0 \geqslant \underline{v}(0)$. We now detail the computation of $\mathcal{N}(\underline{u}, \underline{v})$ in the following subsections by splitting the computations in three zones concerning $x + ct + \xi_0$.

4.1. $x + ct + c\xi_0 < -L_0$. — In this zone, one has necessarily $x + ct + \xi_0 - \xi(t) < -L_0$ and also $x - ct - \xi_0 + \xi(t) < -L_0$ (by asking $2\xi_0 \geqslant 1$). As a consequence, in this zone we have $\mu\phi, \psi, \underline{v} \leqslant \theta/2$ and $\mu\widetilde{\phi}, \widetilde{\psi} \geqslant (1 + \theta_1)/2$. Also $\min(\Gamma, \widetilde{\Gamma}) \equiv \Gamma \equiv e^{\alpha(x + ct + \xi_0 - \xi(t) + L_0)}$ will be denoted $e^{\alpha(\cdots)}$ from now on. Then

$$\mathcal{N}\left(\frac{\underline{u}}{\underline{v}}\right)_1 = -c\dot{\xi}\phi_x + c\dot{\xi}\widetilde{\phi}_x - \frac{\dot{q}_u}{\mu}\widetilde{\phi}_x e^{\alpha(\cdots)} - \frac{q_u}{\mu}c\alpha e^{\alpha(\cdots)} + \frac{q_u}{\mu}c\alpha\dot{\xi}e^{\alpha(\cdots)} + \frac{q_u}{\mu}\alpha^2 e^{\alpha(\cdots)}$$

$$- q_u e^{\alpha(\cdots)} + q_v e^{\alpha(\cdots)}$$

$$\leqslant -\frac{q_u}{\mu}e^{\alpha(\cdots)}\left(-\omega + c\alpha(1 - \dot{\xi}) - \alpha^2\right) + (q_v - q_u)e^{\alpha(\cdots)}.$$

Both terms are already negative thanks to the conditions stated in Section 2. Then a computation similar to the one in the previous section – thus not detailed here –

leads to

$$\mathcal{N}\begin{pmatrix}\underline{u} \\ \underline{v}\end{pmatrix}_2 \leqslant -q_v e^{\alpha(\cdots)}\big(-\omega + c\alpha(1 - \dot{\xi}) - d/D\alpha^2 - \kappa\big) + f(\widetilde{\psi}).$$

This quantity can be made negative provided $\omega \leqslant 2\alpha c$ (which is already the case): indeed, using the exponential decay of $f(\widetilde{\psi})$ in this zone, the above expression can be factorized as $-q_v e^{\alpha(\cdots)} \times (\cdot)$ with $(\cdot)$ having the sign of $-\omega + c\alpha(1 - \dot{\xi}) - d/D\alpha^2 - \kappa \geqslant 0$ provided only that ξ_0 is large enough (but depending on the initial data only through α_0).

4.2. $x + ct + \xi_0 \in (-L_0, L_0)$. — First, we ensure $c\xi \leqslant 1$ by asking that $cB\varepsilon/\omega \leqslant 1$ so by taking $\varepsilon_0 \leqslant \omega/cB$. As a consequence, $x + ct + \xi_0 - c\xi(t) \in (-L_0 - 1, L_0)$ and $x - ct - \xi_0 + c\xi(t) < -L_0$. Since $\omega < 2\alpha c$, the computations of Section 3.2.3 still hold by enlarging the constant B enough.

4.3. $x + ct + \xi_0 > L_0$. — Here three subcases can appear concerning $x - ct - \xi_0$. By exchanging ϕ, ψ and $\widetilde{\phi}, \widetilde{\psi}$ and since $\alpha < \lambda$, the cases $x - ct - \xi_0 \in (-L_0, L_0)$ and $x - ct - \xi_0 > L_0$ are already covered by the computations above. Only the case $x - ct - \xi_0 < -L_0$ remains. In this zone, $x - ct - \xi_0 + c\xi(t) < -L_0 + 1$, so here $\min(\Gamma, \widetilde{\Gamma}) \equiv 1$ and both ψ and $\widetilde{\psi}$ are close to 1.

Observe that the computations of section 2 still hold by splitting this zone in two subzones: $x < 0$ and $x > 0$. In the first one, one will bound $f(\psi) + f(\widetilde{\psi}) - f(\underline{v})$ by $\mathrm{Lip}\, f(1 - \psi) - \beta(1 - \widetilde{\psi} + q_v)$ and in the second one by $\mathrm{Lip}\, f(1 - \widetilde{\psi}) - \beta(1 - \psi + q_v)$. Then, since $\omega < \min(\lambda, \widetilde{\lambda})c$ there holds

$$\mathcal{N}\begin{pmatrix}\overline{u} \\ \overline{v}\end{pmatrix}_2 \leqslant -q_v \times (\cdot),$$

with $(\cdot)$ being positive provided ξ_0 is large enough. This proves Theorem 4.1.   □

5. Initial data with $O(1)$ compact support

We now go back to the original Equation (3) and state the following: if v_0 is large enough, independently on D, then after a waiting time large enough that depends on D, (u, v) falls in the assumptions of the previous theorem.

Theorem 5.1. — *Let L be large enough (independently of D). There exist $M', \delta' > 0$ independent of $D > d$ such that if the initial data of (3) satisfies*

$$v_0 > 1 - \delta' \quad for\ x \in (-M', M')$$

then after a finite time $t_D = D^{1/2}h(D) + O(1)$ one has μu and v satisfying the assumptions of Theorem 4.1, i.e.,

$$\mu u, v \geqslant 1 - \delta \quad for\ all\ x \in (-M\sqrt{D}, M\sqrt{D})$$

As a consequence, starting from the time $t = t_D$, propagation occurs as described in Theorem 4.1.

We will divide the proof in several steps.

Step 1. — Since $D > d$ is ought to be large, u should be very small for small times. Thus we first investigate the equation for v in (3) where u is replaced by 0, and we not only expect to use its solution as a subsolution but we really expect that it will reflect the dynamics of the full solution for some time:

(23)

$$d\partial_y \underline{v} + \underline{v} = 0$$

$$\partial_t \underline{v} - d\Delta \underline{v} = f(\underline{v})$$

$$\partial_y \underline{v} = 0$$

Step 2. — Let $p(y)$ be the largest y-dependent steady solution to (23). The traveling wave for (23) connecting 0 and $p(y)$ will serve to build a subsolution for (23) propagating just as in Theorem 4.1 but here at speed $c_p = O(1)$. This will give a lower bound on the boundary data $v(x, -L) \geqslant \underline{v}(x, -L) \geqslant \cdots$. This will be the purpose of Lemma 5.2.

Step 3. — Using this lower bound, we then go back to (4): we show that even without the reaction term, this lower bound suffices to have $\mu u, v \geqslant 1 - \delta$ on $(-M, M)$ within a finite time t_D. As a consequence, this is the case also for the nonlinear problem. This will be proved in a final step. Observe that here we use $f \geqslant 0$. If we were looking for instance at a bistable nonlinearity this would still be true but we would need to add a positive zero-order term in these computations.

We recall the following elementary fact, that we will freely use in the sequel.

Lemma 5.1. — *There exists $L_0 > 0$ such that if $L > L_0$, there exists a solution $p(y)$ to*

(24a)
(24b)
(24c)
$$\begin{cases} -dp'' = f(p) \\ p'(-L) = 0 \\ dp'(0) + p(0) = 0 \end{cases}$$

with $p > 0$ concave decreasing and $p(0) \geqslant p(-L) = 1 - \delta'' > 1 - \delta$. Moreover $\delta'' \to 0$ as $L \to +\infty$.

Let us now prove the following:

Lemma 5.2. — *Let v be a solution of (23). There exist $\delta', M' > 0$ independent of D such that if $v_0 > 1 - \delta'$ for $x \in (-M', M')$, there holds*

$$v(t, x, -L) \geqslant (1 - \delta'')\varphi_t(x) - Ce^{-bt},$$

where $C, b > 0$ do not depend on D and (φ_t) is bounded in $\mathcal{C}^3$ such that $\varphi_t(x) = 1$ for $|x| \leqslant \frac{c_p}{2}t$ and $\varphi_t(x) = 0$ for $|x| \geqslant c_p t$ for some speed $c_p > 0$ independent of D.

Proof. — First, that there exists a traveling wave solution of (23) with speed $c_p > 0$ independent of D connecting 0 and $p(y)$ has to be established: for this we refer to Berestycki-Nirenberg [5] which gives the existence of an increasing (in x) traveling front $\psi(x, y)$ with exponential convergence towards 0 and $p(y)$ as $x \to \pm\infty$.

Now we notice that the subsolution argument in Theorem 4.1 can be used but in a simpler fashion for the Robin homogeneous boundary value problem (23): on the one hand, the structure of the problem is simpler than the one studied in Theorem 4.1 since here we deal with a single equation, the original construction of [15] with $q_v = \varepsilon e^{-\omega t}$ will suffice. On the other hand, 1 is not a steady state for the problem so one has to replace 1 by $p(y)$ in the computations. Nonetheless, one can check that the above computations still hold with the suitable subsolution

$$\psi + \widetilde{\psi} - p - q_v \min(\Gamma, \widetilde{\Gamma}).$$

As a consequence, just as in Theorem 4.1, provided v_0 is above an initial shift of a pair of waves – hence the existence of δ' and M' – its level lines will be pushed by below by the pair of waves traveling as $\pm c_p t \mp O(1)$. This implies the desired bound.   □

End of the proof of Theorem 5.1. — Let (u^D, v^D) be the solution of (4) starting from compactly supported $0 \leqslant \mu u_0, v_0 \leqslant 1$ and let v_0 satisfy the rescaled assumptions of Theorem 5.1. First, let $h(D)$ be any positive function such that $h(D)$ grows to infinity as $D \to \infty$, and set $T_D = D^{1/2}h(D)$. We now show the following

$$(25) \quad \liminf_{t \to +\infty} \inf_{D > d} \min_{(x,y) \in \overline{\Omega}_{L,M}} \{\mu u^D(T_D + t, x), v^D(T_D + t, x, y)\} \geqslant 1 - \delta'' > 1 - \delta,$$

where $\Omega_{L,M} = (-M, M) \times (-L, 0)$. First, it is an easy but tedious exercise to see that the left hand-side of (25) can be characterized as the limit as $n \to +\infty$ of some $\mu u^{D_n}(T_{D_n} + t_n, x_n)$ or $v^{D_n}(T_{D_n} + t_n, x_n, y_n)$, where $t_n \to +\infty$, $D_n > d$, $(x_n, y_n) \in \overline{\Omega}_{L,M}$. We then extract from (t_n, D_n, x_n, y_n) a subsequence so that $x_n \to x_\infty$ and $y_n \to y_\infty$. Our objective is to extract from (u, v) a subsequence converging to some limiting (u_∞, v_∞) to which the maximum principle will apply and force the above limit to be $\geqslant 1 - \delta''$. The difficulty comes from the fact that (D_n) might be unbounded, so that standard parabolic estimates and the usual maximum principle might fail at the limit. Two cases can appear.

Case 1. (D_n) is unbounded. — Then we extract again so that $D_n \to +\infty$. Let

$$u_n(t, x) := u^{D_n}(t_{D_n} + t_n + t, x_\infty + x), \quad v_n(t, x, y) := v^{D_n}(t_{D_n} + t_n + t, x_\infty + x, y).$$

Since $f \geqslant 0$ and thanks to Lemma 5.2 above, the comparison principle implies that $(u_n, v_n) \geqslant (\underline{u}_n, \underline{v}_n)$, where $(\underline{u}_n, \underline{v}_n)$ is the solution of

$$(26) \quad \begin{array}{c} \partial_t \underline{u}_n - \partial_{xx}^2 \underline{u}_n = \underline{v}_n - \mu \underline{u}_n \\ \hline d\partial_y \underline{v}_n = \mu \underline{u}_n - \underline{v}_n \\ \partial_t \underline{v}_n - \dfrac{d}{D_n} \partial_{xx}^2 \underline{v}_n - d\partial_{yy}^2 \underline{v}_n = 0 \\ \hline \underline{v}_n = (1 - \delta'')\varphi_{T_{D_n} + t_n}(x_\infty + x) - Ce^{-b(t_n + t)} \end{array}$$

with any initial data compatible with the boundary conditions and below the one of (u_n, v_n).

Since $d/D_n \to 0$, the standard parabolic estimates (see [16, Chap. 6]) applied to v_n will fall concerning the x-derivatives. We overcome this difficulty since Equation (26) is linear and the boundary data $\underline{v}_n(t, x, -L)$ is bounded in $\mathcal{C}^3$: the maximum principle applied to x-derivatives of (u_n, v_n) up to order 3 gives that they are all bounded independently of n:

$$|\partial^2_{xx}\underline{u}_n|_\infty, \ |\partial^3_{xxx}\underline{u}_n|_\infty, \ |\partial^2_{xx}\underline{v}_n|_\infty, \ |\partial^3_{xxx}\underline{v}_n|_\infty \leqslant C_1$$

Now concerning the y-derivatives, even though $d/D_n \to 0$ the standard estimates hold: indeed since $\underline{v}_n \leqslant 1$, standard L^p parabolic estimates (see [16, Chap. 9.5]) with p large enough applied to $\underline{u}_n$ give that $\underline{u}_n$ is bounded in $\mathcal{C}^{\alpha,1+\alpha}$ by some C_2. Now rescale by $x \leftarrow x\sqrt{D_n}$ so that $|\underline{u}_n(t, x/\sqrt{D_n})|_{\mathcal{C}^{\alpha,1+\alpha}} \leqslant C_3$ (the semi-norms of the derivatives even go to zero since $1/D_n \to 0$). Moreover, under this rescaling $-d/D_n\partial^2_{xx} - d\partial^2_{yy}$ becomes $-d\Delta$, so that standard parabolic estimates up to the Robin boundary apply and give that $|\underline{v}_n(t, x/\sqrt{D_n}, y)|_{\mathcal{C}^{1+\alpha/2,2+\alpha}} \leqslant C_4$. Since this rescaling does not impact ∂_y or ∂_t, this gives

$$|\partial_t\underline{v}_n|_{\alpha/2}, \ |\partial_y\underline{v}_n|_\alpha, \ |\partial^2_{yy}\underline{v}_n|_\alpha \leqslant C_4$$

The bound on $\partial^2_{xy}\underline{v}_n$ follows by combining the two arguments above, and finally by plugging the estimate on v in the equation for u, standard Schauder estimates yield that $\underline{u}_n$ is bounded in $\mathcal{C}^{1+\alpha/2,2+\alpha}$. In the end one can extract from $(\underline{u}_n, \underline{v}_n)$ some subsequence converging in $\mathcal{C}^{1,2}_{\text{loc}}$ to some (u_∞, v_∞) global in time (since $t_n \to +\infty$) solving

(27)
$$\begin{array}{c} \underline{\partial_t u_\infty - \partial^2_{xx}u_\infty = v_\infty - \mu u_\infty} \\[4pt] d\partial_y v_\infty = \mu u_\infty - v_\infty \\[8pt] \partial_t v_\infty - d\partial^2_{yy}v_\infty = 0 \\[4pt] \overline{v_\infty = (1 - \delta'')} \end{array}$$

Indeed, $v_\infty(t, x, -L) \equiv 1 - \delta''$ since

$$1 - \delta'' \geqslant \underline{v}_n(t, x, -L) \geqslant 1 - \delta'' - Ce^{-b(t_{D_n}+t_n+t)}$$

for $x \in (-\frac{c_p}{2}h(D_n), \frac{c_p}{2}h(D_n))$ by Lemma 5.2 above and by use of T_{D_n}.

Since (u_∞, v_∞) are global in time, there is no initial data anymore and the maximum principle applies to give

$$\mu u_\infty, v_\infty \equiv 1 - \delta''.$$

Indeed, no value different than $1 - \delta''$ can be reached, because then (u, v) would have an infimum smaller or a supremum larger than $1 - \delta''$. By translating over time (which is possible since the solution is global) this infimum or supremum would become a minimum or maximum, that cannot be reached by u because of the strong parabolic maximum principle, and neither by v by the strong parabolic maximum principle and Hopf's lemma applied to the suitable y-slice.

Case 2. (D_n) *is bounded.* — Then one extracts so that $D_n \to D_\infty > d$ and the above proof is much simpler since standard regularity and maximum principle apply. Moreover T_D is not necessary.

In both cases, the $\liminf$ above is $\geqslant u_\infty(0,0) = 1 - \delta''$ or $\geqslant v_\infty(0,0,y_\infty) = 1 - \delta''$, thus (25) holds. Theorem 5.1 follows easily: indeed, there exists t_1 independent of D such that after $t_D = T_D + t_1$, $\mu u, v > 1 - \delta$ on $(-M, M)$. $\qquad\square$

6. Lower bound on the waiting time

In this subsection, (u, v) denotes the solution of (4). Let $\varepsilon := D^{-1/2}$ and v^0 solve

$$
\begin{array}{c}
\hline
d\partial_y v^0 + v^0 = 0 \\[2mm]
\partial_t v^0 - d\varepsilon^2 \partial_{xx}^2 v^0 - d\partial_{yy}^2 v^0 = f(v^0) \\[2mm]
\partial_y v^0 = 0 \\
\hline
\end{array}
\tag{28}
$$

sharing the same initial data as v: $v^0(0) = v_0$ with $0 \leqslant v_0 \leqslant 1$. Observe that v^0 is the rescaling of the subsolution $\underline{v}$ already introduced in Equation (23). The aim of this subsection is to give an estimate on the time during which v is close to v^0. More precisely we will show the following.

Theorem 6.1. — *Let* $\alpha \in (0,1)$ *and define*

$$
T_{\alpha,\varepsilon} := \sup\{ \ T > 0 \mid |v - v^0| < \varepsilon^\alpha \text{ for all } 0 < t < T \text{ and } (x,y) \in \Omega_L \}.
$$

Then for all $0 < \delta < \min\left(\alpha, 2/7, \frac{2}{5}(1-\alpha)\right)$ *one has*

$$
\lim_{\varepsilon \to 0} \varepsilon^\delta T_{\alpha,\varepsilon} = +\infty.
$$

Remark 6.1. — We would like to insist on the fact that v^0 is not the initial data here but the solution of the above described problem. The limiting case is $\delta = \alpha = 2/7$. Indeed, picking $\alpha = 2/7 + \kappa$, we can chose δ lower and as close to $2/7 - \kappa$ as we want. This theorem thus states that for every small $\kappa > 0$, by waiting a time of order $\varepsilon^{-2/7+\kappa} = D^{1/7-\kappa/2}$ we are still as close as we want to v^0 – which propagates at a speed independent from D – and u is as close as we want to zero. This is Theorem 2.6.

Let us define $w = v - v^0$. Observe that (u, w) solves

$$
\begin{array}{c}
\partial_t u - \partial_{xx}^2 u + \mu u - w = v^0 \\
\hline
d\partial_y w + w = \mu u \\[2mm]
\partial_t w - d\varepsilon^2 \partial_{xx}^2 w - d\partial_{yy}^2 w = f'(\cdot)w \\[2mm]
\partial_y w = 0 \\
\hline
\end{array}
\tag{29}
$$

where $\cdot \in [v, v^0] \cup [v^0, v]$ (by Taylor's formula). The idea of the proof is to decouple Equation (29) by decomposing w in two parts. Let us set $w = w^1 + \overline{w}$, where

$$
(30) \quad
\begin{array}{c}
\hline
d\partial_y w^1 + w^1 = 0 \\[4pt]
\partial_t w^1 - d\varepsilon^2 \partial_{xx}^2 w^1 - d\partial_{yy}^2 w^1 = f'(\cdot)w \\[4pt]
\partial_y w^1 = 0 \\
\hline
\end{array}
$$

and

$$
(31) \quad
\begin{array}{c}
\hline
\partial_t u - \partial_{xx}^2 u + \mu u - \overline{w} = w^1 + v^0 \\[4pt]
d\partial_y \overline{w} + \overline{w} = \mu u \\[4pt]
\partial_t \overline{w} - d\varepsilon^2 \partial_{xx}^2 \overline{w} - d\partial_{yy}^2 \overline{w} = 0 \\[4pt]
\partial_y \overline{w} = 0 \\
\hline
\end{array}
$$

Observe that $T_{\varepsilon,\alpha}$ exists by continuity. We now work by contradiction to show that if $T_{\alpha,\varepsilon} = (1/\varepsilon)^\delta$, then $|w|$ stays of order less than $\varepsilon^{\alpha'}$ with $\alpha' > \alpha$. During the rest of the proof, this will be abbreviated with "$\ll \varepsilon^\alpha$".

The scheme is as follows. First, we derive an L^1 estimate on w^1 by Duhamel's formula. This, inserted in estimate (51), yields the desired estimate on u and then on $\overline{w}$. We then go back to w^1 to obtain the desired estimate, by a more intricate supersolution argument.

6.1. L^1 BOUND ON w^1. — By definition, up to time $T_{\varepsilon,\alpha}$ one has $|w| \leqslant \varepsilon^\alpha$. By Duhamel's formula and the maximum principle for Equation (30), this yields

$$(32) \qquad |w^1| \leqslant \varepsilon^\alpha \int_0^t e^{d\varepsilon^2(t-s)\partial_{xx}} e^{d(t-s)\partial_{yy}^{NR}} f'(\cdot) \mathrm{d}s,$$

where ∂_{yy}^{NR} denotes ∂_{yy} endowed with the Neuman-Robin boundary condition of (30). Since $\cdot \leqslant v^0 + \varepsilon^\alpha$, using the above results and rescaling them, one knows that $v^0 \leqslant \theta$ for $x \leqslant (a + c_p t)\varepsilon$ for some constant $a > 0$. As a consequence,

$$|f'(\cdot)| \leqslant \operatorname{Lip} f \times \mathbf{1}_{((-a-c_p s)\varepsilon, (a+c_p s)\varepsilon)}.$$

Also, by the maximum principle, there exist $C(d) > 0$ and $\lambda_1(d) > 0$ such that

$$e^{d(t-s)\partial_{yy}^{NR}} \mathbf{1}_{((-a-c_p s)\varepsilon, (a+c_p s)\varepsilon)} \leqslant C e^{-\lambda_1(t-s)} \mathbf{1}_{((-a-c_p s)\varepsilon, (a+c_p s)\varepsilon)}.$$

Using both estimates in (32), the maximum principle yields

$$(33) \qquad |w^1| \leqslant C \operatorname{Lip} f \varepsilon^\alpha \int_0^t e^{d\varepsilon^2(t-s)\partial_{xx}} e^{-\lambda_1(t-s)} \mathbf{1}_{((-a-c_p s)\varepsilon, (a+c_p s)\varepsilon)} \mathrm{d}s.$$

Since $e^{d\varepsilon^2(t-s)\partial_{xx}}$ preserves the L^1 norm, this gives the estimate, for some constants C_1, C_2 that do not depend on ε:

$$(34) \quad |w^1(\cdot, y)|_{L^1(\mathbb{R})} \leqslant C \operatorname{Lip} f \varepsilon^\alpha \int_0^t e^{-\lambda_1(t-s)} 2(c_p s + a)\varepsilon \mathrm{d}s = \varepsilon^{\alpha+1}(C_1 + C_2 t) \ll \varepsilon^\alpha,$$

since $\delta < 1$.

6.2. **Estimate on u and $\overline{w}$.** — Using the appendix estimate (51) and Duhamel's formula one gets

$$(35) \qquad |u| \leqslant C_3 \int_0^t \left(\int_{\mathbb{R}} \frac{e^{-(x'-x)^2/(4a(t-s))}}{\sqrt{4\pi a(t-s)}} |w^1 + v^0|(s, x', 0) \mathrm{d}x' \right) \mathrm{d}s$$

$$(36) \qquad\qquad + C_4 \int_0^t e^{-\omega(t-s)} |(w^1 + v^0)(s, \cdot, 0)|_{L^1(\mathbb{R})} \mathrm{d}s.$$

First observe that due to the above results and the rescaling, one has

$$|v^0(s, \cdot, y)|_{L^1} \leqslant (C_5 + c_p s)\varepsilon.$$

Using this and estimate (34), we deal with the second term:

$$(36) \leqslant C_4 \int_0^t e^{-\omega(t-s)} \left(\varepsilon^{\alpha+1}(C_1 + C_2 s) + \varepsilon(C_5 + c_p s) \right) \mathrm{d}s \leqslant (C_6 + C_7 t)(\varepsilon^{\alpha+1} + \varepsilon) \ll \varepsilon^\alpha$$

since $1 - \delta > \alpha$. We now deal with the first term:

$$(35) \leqslant C_3 \int_0^t \frac{1}{\sqrt{4\pi a(t-s)}} |(w^1 + v^0)(s, \cdot, 0)|_{L^1(\mathbb{R})} \mathrm{d}s \leqslant (C_8 t^{1/2} + C_9 t^{3/2})(\varepsilon^{\alpha+1} + \varepsilon) \ll \varepsilon^\alpha$$

since $1 - \frac{3}{2}\delta > \alpha$. As a consequence, $|u| \ll \varepsilon^\alpha$. Now seeing Equation (31) as a boundary value problem for $\overline{w}$, we see that the above estimate on μu provides an easy supersolution that stays above $\overline{w}$, that is

$$\overline{w} \leqslant \mu \left(C_6 + C_7 t + C_8 t^{1/2} + C_9 t^{3/2} \right) \left(\varepsilon^{\alpha+1} + \varepsilon \right) \ll \varepsilon^\alpha.$$

6.3. **Back to w^1.** — Using $w = w^1 + \overline{w}$ we rewrite Equation (30) as a linear non-homogeneous problem:

$$(37) \qquad \begin{array}{c} \hline \\ d\partial_y w^1 + w^1 = 0 \\[4pt] \partial_t w^1 - d\varepsilon^2 \partial_{xx}^2 w^1 - d\partial_{yy}^2 w^1 - f'(\cdot)w^1 = f'(\cdot)\overline{w} \\[4pt] \partial_y w^1 = 0 \\ \hline \end{array}$$

Since $w^1(0) = 0$, Duhamel's formula gives

$$(38) \qquad w^1(t) = \int_0^t w_h^s(t-s) \mathrm{d}s$$

where w_h^s solves

$$(39) \qquad \begin{array}{c} \hline \\ d\partial_y w_h^s + w_h^s = 0 \\[4pt] \mathcal{L}w_h^s := \partial_t w_h^s - d\varepsilon^2 \partial_{xx}^2 w_h^s - d\partial_{yy}^2 w_h^s - f'(\cdot)w_h^s = 0 \\[4pt] \partial_y w_h^s = 0 \\ \hline \end{array}$$

along with the initial condition $w_h^s(0) = (f'(\cdot)\overline{w})(s)$. Inspired by the linearization and a rescaling of the supersolution to the non-linear equation in the previous section, and using the same notation as in it, we look for a supersolution with the form

$$\overline{w_h^s}(t) = \xi(t)\partial_x\psi(x + c_p\varepsilon t + x_0^s) - \xi(t)\partial_x\widetilde{\psi}(x - c_p\varepsilon t - x_0^s) + Ce^{-\omega t}\min(\Gamma, \widetilde{\Gamma})$$

for some $\xi(t)$ increasing in time and initial shift x_0^s. First of all, we need to ensure ordering of the initial data. That is: $\overline{w_h^s}(0) \geqslant (f'(\cdot)\overline{w})(s)$, which is obtained provided

$$\overline{w_h^s}(0) \geqslant \operatorname{Lip} f \varepsilon^{\alpha+\nu} \mathbf{1}_{((-a-c_ps)\varepsilon,(a+c_ps)\varepsilon)}$$

for some $\nu < 1 - 3/2\delta - \alpha$ (since $\overline{w} \leqslant K\varepsilon^{1-\frac{3}{2}\delta}$ thanks to the computations above). We achieve this by asking

$$(40) \qquad C = (\operatorname{Lip} f)\varepsilon^{\alpha+\nu}, \quad x_0^s > (a + c_p)s\varepsilon + L_0.$$

We will also see below that we need $\delta < \nu$. Combining both these conditions imposes $\delta < 1 - \frac{3}{2}\delta - \alpha$, i.e., the assumption $\delta < \frac{2}{5}(1 - \alpha)$ of Theorem 6.1.

Now, straightforward computations give

$$\mathcal{L}\overline{w_h^s} = \xi f'(\psi)\partial_x\psi - \xi f'(\widetilde{\psi})\partial_x\widetilde{\psi} + \dot{\xi}\partial_x\psi - \dot{\xi}\partial_x\widetilde{\psi} - f'(\cdot)(\xi\partial_x\psi - \xi\partial_x\widetilde{\psi})$$
$$- \omega Ce^{-\omega t}\min(\Gamma, \widetilde{\Gamma}) + Ce^{-\omega t}\partial_t\min(\Gamma, \widetilde{\Gamma})$$
$$+ Ce^{-\omega t}\partial_{xx}\min(\Gamma, \widetilde{\Gamma}) - f'(\cdot)Ce^{-\omega t}\min(\Gamma, \widetilde{\Gamma}).$$

As in Section 3.2, we analyze the sign of this quantity in three separate zones. Observe that due to the rescaling between (3) and (4), decay exponents Θ, Θ_0 (resp. the α and α_0 from Sec. 3.2) and $\lambda, \widetilde{\lambda}$ scale here as $1/\varepsilon$ as well as the lower bounds on the derivatives δ_L. Remember also that we are looking only at times $t \leqslant \varepsilon^{-\delta}$, so we only need to find a supersolution up to this time. We also reinitialize the constants C_i and K_i which will be positive constants independent of ε.

6.3.1. $x + c_p\varepsilon t + x_0^s < -L_0$. — As before, in this zone we have $\psi \leqslant \theta/2, \widetilde{\psi} \geqslant (1+\theta_1)/2$, $\psi, \partial_x\psi \leqslant C_1 e^{-\lambda(x+c_p\varepsilon t+x_0^s+L_0)}$, $1 - \widetilde{\psi}, \partial_x\widetilde{\psi} \leqslant C_1 e^{-\widetilde{\lambda}(x-c_p\varepsilon t-x_0^s+L_0)}$ and $\min(\Gamma, \widetilde{\Gamma}) \equiv \Gamma = e^{\Theta(x+c_p\varepsilon t+x_0^s)} =: e^{\Theta(\cdots)}$. As a consequence and since $\Theta < \widetilde{\lambda}$, one has

$$\mathcal{L}\overline{w_h^s} \geqslant \dot{\xi}\partial_x\psi - \dot{\xi}\partial_x\widetilde{\psi}$$
$$+ \operatorname{Lip} f \varepsilon^{\alpha+\nu} e^{-\omega t} e^{\Theta(\cdots)}\left[\left(-\omega + c_p\Theta\varepsilon + \Theta^2\right) - \frac{\xi}{\varepsilon^{\alpha+\nu}}e^{(\omega-2\Theta c_p\varepsilon)t}e^{-2x_0^s\Theta}\right].$$

The first term inside the brackets is positive provided $\omega < \Theta^2$ (which is not a constraint since Θ grows as $1/\varepsilon$) and we can make the whole bracket positive provided

$$(41) \qquad \omega < 2\Theta c_p\varepsilon$$

(observe that the right-hand side in (41) is bounded from above and below by positive constants that do not depend on ε) by taking

$$(42) \qquad x_0^s > -\frac{1}{2\Theta}\ln\left(\left(-\omega + c_p\Theta\varepsilon + \Theta^2\right)\frac{\varepsilon^{\alpha+\nu}}{\xi(\varepsilon^{-\delta})}\right).$$

This will be a constraint on our future choice of $\xi(t)$, to be kept in mind.

6.3.2. $x + c_p \varepsilon t + x_0^s \in (-L_0, L_0)$. — Here $x - c_p \varepsilon t - x_0^s < -L_0$, and $\widetilde{\psi} \geqslant (1 + \theta_1)/2$, $1 - \widetilde{\psi}, \partial_x \widetilde{\psi} \leqslant C_2 e^{-\widetilde{\lambda}(x - c_p \varepsilon t - x_0^s + L_0)}$, $f'(\widetilde{\psi}) \leqslant -\beta$ and $\min(\Gamma, \widetilde{\Gamma}) = \Gamma$.

$$\mathcal{L}\overline{w_h^s} \geqslant \xi f'(\psi)\partial_x \psi - \xi f'(\widetilde{\psi})\partial_x \widetilde{\psi} + 2\delta_{2L_0}\dot{\xi} - \varepsilon^{\alpha+\nu}e^{-\omega t}\left(\omega + |\Gamma|_{\mathcal{C}^2}\right)$$
$$- f'(\cdot)\left[\xi\partial_x\psi - \xi\partial_x\widetilde{\psi}\right] - f'(\cdot)\varepsilon^{\alpha+\nu}e^{-\omega t}\Gamma$$
$$\geqslant \xi f'(\psi)\partial_x\psi - \xi f'(\widetilde{\psi})\partial_x\psi + 2\delta_{2L_0}\dot{\xi} - \varepsilon^{\alpha+\nu}e^{-\omega t}\underbrace{\left(\omega + |\Gamma|_{\mathcal{C}^2} + \mathrm{Lip}(f')|\Gamma|_{\mathcal{C}^2}\right)}_{C_3}$$
$$- f'(\cdot)\left[\xi\partial_x\psi - \xi\partial_x\widetilde{\psi}\right].$$

We make this positive by counterbalancing the negative terms thanks to $\delta_{2L_0}\dot{\xi}$, by asking

$$(43) \qquad\qquad \dot{\xi} \geqslant \frac{2}{\delta_{2L_0}}\varepsilon^{\alpha+\nu}e^{-\omega t}C_3 =: K_1 \varepsilon^{\alpha+\nu+1}e^{-\omega t}.$$

Moreover by using $\Theta < \widetilde{\lambda}$, $\omega < 2\Theta c_p \varepsilon$ and the previous expression of $e^{-2\Theta x_0^s}$ one obtains

$$-\xi f'(\cdot)\partial_x\widetilde{\psi}, \ -\xi f'(\widetilde{\psi})\partial_x\widetilde{\psi} \geqslant -\varepsilon^{\alpha+\nu}C_4 e^{-\omega t},$$

so that we ask also for

$$\dot{\xi} \geqslant K_2 \varepsilon^{\alpha+\nu+1}e^{-\omega t},$$

which is implied by (43) by taking K_1 large enough. The last term to counterbalance is

$$\xi\partial_x\psi[f'(\psi) - f'(\cdot)] \geqslant -\xi\partial_x\psi\,\mathrm{Lip}(f)\left(|v_0 - \psi| + \varepsilon^\alpha\right)$$

by the triangle inequality and the definition of $\cdot$. But we also know that

$$|\psi - v_0| \leqslant |\psi - (\psi + \widetilde{\psi} - 1)| + |(\psi + \widetilde{\psi} - 1) - v_0| \leqslant |1 - \widetilde{\psi}| + C_5 e^{-\omega_0 t}$$

by Section 5 for some $C_5, \omega_0 > 0$ independent of ε. Now just as above, one can use the exponential decay of $1 - \widetilde{\psi}$ in the current zone to prove that there exists $C_6 > 0$ such that $|1 - \widetilde{\psi}| \leqslant C_6 e^{-\omega t}$. In the end

$$\xi\partial_x\psi[f'(\psi) - f'(\cdot)] \geqslant -\mathrm{Lip}(f)\xi\partial_x\psi\left[C_6 e^{-\omega t} + C_5 e^{-\omega_0 t} + \varepsilon^\alpha\right].$$

We can reduce ω and change the constants so that

$$\xi\partial_x\psi[f'(\psi) - f'(\cdot)] \geqslant -\mathrm{Lip}(f)\xi\partial_x\psi\left[C_7 e^{-\omega t} + \varepsilon^\alpha\right]$$

and we counterbalance this term by asking (remember the additional power of ε factor due to the scaling of δ_{2L_0})

$$(44) \qquad\qquad \dot{\xi} \geqslant K_2 \varepsilon \xi e^{-\omega t}, \quad \dot{\xi} \geqslant K_3 \xi \varepsilon^{\alpha+1}.$$

We now have to find a suitable increasing function $t \mapsto \xi(t)$ satisfying (43), (44) and that should not increase too much so that $w^1(\varepsilon^{-\delta}) \ll \varepsilon^\alpha$. Since the order between the right-hand sides in (44) changes at some point in time, we define ξ in two parts as a continuous but only piecewise $\mathcal{C}^1$ function. This is not a problem since one can apply the maximum principle a second time starting from the junction. We propose

$$\xi(t) := \begin{cases} \varepsilon^{\alpha+\nu}e^{-K_2(\varepsilon/\omega)e^{-\omega t}} & \text{if } t < \frac{1}{\omega}\ln\left(\frac{K_3}{K_2}\varepsilon^{-\alpha}\right) \\ B(\varepsilon)\varepsilon^{\alpha+\nu}e^{K_3\varepsilon^{\alpha+1}t} & \text{if } t \geqslant \frac{1}{\omega}\ln\left(\frac{K_3}{K_2}\varepsilon^{-\alpha}\right), \end{cases}$$

with $B(\varepsilon) > 0$ uniformly bounded from above and by below in ε is chosen so that ξ is continuous:

$$B(\varepsilon) = e^{K_3 \varepsilon^{\alpha+1}(1/\omega)\ln\left(\frac{K_3}{K_2}\varepsilon^{-\alpha}\right)} e^{K_2(\varepsilon/\omega)\frac{K_2}{K_3}\varepsilon^\alpha}.$$

Observe that (44) is automatically satisfied since we just integrated the stronger differential equation between the two on the associated time-intervals. Observe that (43) is indeed satisfied provided $K_2, K_3 > K_1$ and ε is small.

Now with this choice of ξ – since $\xi \ll \varepsilon^{\alpha+\nu}$ up to time $\varepsilon^{-\delta}$ – observe that the remaining condition on x_0^s (42) is void since $x_0^s > 0$ and it reduces to the initial one (40). Finally, the only condition on ω is (41).

6.3.3. $x + c_p\varepsilon t + x_0^s \geqslant L_0$. — As before, we deal with this last zone by using symmetry and by repeating the arguments above: no stronger condition appears and the computations above hold (by eventually changing the constants).

6.4. End of the proof of Theorem 6.1. — We now estimate w_1 thanks to the super-solution. Coming back to (38) with $t = \varepsilon^{-\delta}$ one gets

$$w_1(t) \leqslant K_6 \int_0^t \xi(s)\mathrm{d}s + K_7 \int_0^t \varepsilon^{\alpha+\nu} e^{-\omega s}\mathrm{d}s.$$

The second term is bounded by $K_7\varepsilon^{\alpha+\nu}/\omega \ll \varepsilon^\alpha$. For the first term, we divide the integral in two parts. Call

$$t_j = \frac{1}{\omega}\ln\left(\frac{K_3}{K_2}\varepsilon^{-\alpha}\right)$$

the junction time.

$$\int_0^t \xi(s)\mathrm{d}s \leqslant \int_0^{t_j} \xi(s)\mathrm{d}s + \int_{t_j}^t \xi(s)\mathrm{d}s \leqslant K_8\varepsilon^{\alpha+\nu}\ln(\varepsilon^{-\alpha}) + \frac{B\varepsilon^{\alpha+\nu}}{K_3\varepsilon^{\alpha+1}}\left[e^{K_3\varepsilon^{\alpha+1}t} - 1\right]$$

by using a crude upper bound for the first part in the definition of $\xi(t)$. Now since $\delta < \alpha + 1$,

$$\frac{B\varepsilon^{\alpha+\nu}}{K_3\varepsilon^{\alpha+1}}\left[e^{K_3\varepsilon^{\alpha+1}t} - 1\right] \underset{\varepsilon\to 0}{\sim} B\varepsilon^{\alpha+\nu}t = B\varepsilon^{\alpha+\nu-\delta} \ll \varepsilon^\alpha$$

because $\nu > \delta$. In the end, both $\overline{w}$ and w_1 are $\ll \varepsilon^\alpha$ up to time $t = \varepsilon^{-\delta}$ so we have a contradiction.

7. Initial data supported on the road only

In this section we investigate the behaviour of solutions starting from $(u_0, v_0) = (\mathbf{1}_{(-a,a)}, 0)$. We still denote $\varepsilon := 1/\sqrt{D}$.

7.1. a is small

Theorem 7.1. — *There exists $a_0 > 0$ such that for $a < a_0$, the solution of (4) starting from $(\mathbf{1}_{(-a,a)}, 0)$ decays to 0 uniformly.*

The proof relies on a suitable reformulation of Equation (3) and a crude bound on f. Observe that, if we replace v by its even extension on $\mathbb{R} \times [-L, L]$, we have

$$(45) \qquad \begin{cases} \partial_t v - d\varepsilon^2 \partial_{xx}^2 v - d\partial_{yy}^2 v = f(v) + 2d(\mu u - v)(x, 0)\mathrm{d}\lambda_{y=0} \\ \partial_t u - \partial_{xx}^2 u + \mu u = v(x, 0) \end{cases}$$

where $\mathrm{d}\lambda_{y=0}$ denotes the Lebesgue measure on the line $\{y = 0\}$.

Lemma 7.1. — *Let $C = \max(\mathrm{Lip}\, f, 2d)$. Then*

$$v(t, x, y) \leqslant C(t + 2C'\sqrt{t}),$$

where C' is a constant that depends only on d and L.

Proof. — This is basically an Aronson type inequality (see [1]), we give a quick computation here. By Duhamel's formula,

$$(46) \qquad v(t, x, y) = \int_0^t e^{s\Delta_{d,\varepsilon}^N}[f(v(t - s, x, y)) + 2d(\mu u - v)(t - s, x, 0)\mathrm{d}\lambda_{y=0}]\mathrm{d}s,$$

where $\Delta_{d,\varepsilon}^N = d\varepsilon^2 \partial_{xx}^2 + d\partial_{yy}^2$ endowed with Neumann boundary conditions on $y = \pm L$. Since $\mathbb{R} \times (-L, L)$ is a product domain and since ∂_{xx}^2 and ∂_{yy}^2 commute, we can compute this heat kernel as follows. Denote $\lambda_k = d(k\pi/(2L))^2$ the eigenvalues of $d\partial_{yy}^2$ on $(-L, L)$ with Neumann conditions and ϕ_k the associated eigenfunctions. Then the heat kernel is

$$K_2(t, y, y') = \sum_{k \geqslant 0} e^{-\lambda_k t}\phi_k(y)\phi_k(y').$$

For $d\varepsilon^2 \partial_{xx}^2$ on $\mathbb{R}$ the heat kernel is

$$K_1(t, x, x') = \frac{1}{\sqrt{4\pi d\varepsilon^2 t}}\, e^{-(x-x')^2/(4d\varepsilon^2 t)}.$$

As a consequence,

$$e^{s\Delta_{d,\varepsilon}^N}\mathrm{d}\lambda_{y=0} = \int_{\mathbb{R}} K_1(s, x, x')K_2(s, y, 0)\mathrm{d}x' = \sum_{k \geqslant 0} e^{-\lambda_k s}\phi_k(y)\phi_k(0),$$

which of course depends only on y and is even in y (the ϕ_k being even or odd). Observe that this is nothing more than the fundamental solution of the diffusion equation in y on $(-L, L)$. Because the ϕ_k are uniformly bounded by C' depending only on d and L one gets

$$e^{s\Delta_{d,\varepsilon}^N}\mathrm{d}\lambda_{y=0} \leqslant C' \sum_{k \geqslant 0} e^{-\lambda_k s} \leqslant C'/\sqrt{s}$$

for another constant C'. The last inequality comes from the growth of λ_k as Ck^2.

Going back to (46) and using $f(v) \leqslant \mathrm{Lip}\, f$ as well as $\mu u - v \leqslant 1$ and the positivity of the integral, one gets

$$v(t, x, y) \leqslant C \int_0^t (1 + C'/\sqrt{s})\mathrm{d}s \leqslant C(t + 2C'\sqrt{t}),$$

which implies the lemma. □

Lemma 7.2. — *We have*

$$u(t, x) \leqslant \frac{2e^{-\mu t}}{\sqrt{4\pi t}}\, a + \frac{Ct^2}{2} + \frac{4CC'}{3}\, t^{3/2}.$$

Proof. — We insert the previous estimate on $v(t, x, 0)$ in the equation satisfied by u and solve it using Duhamel's formula. By the maximum principle, this gives the following upper bound:

$$u(t, x) \leqslant e^{-\mu t} e^{t\partial_{xx}^2} u_0 + C \int_0^t e^{-\mu(t-s)} e^{(t-s)\partial_{xx}^2} (s + 2C'\sqrt{s}) \mathrm{d}s$$

$$= e^{-\mu t} \int_{-a}^{a} \frac{1}{\sqrt{4\pi t}} e^{-(x-x')^2/4t} \mathrm{d}x' + C\Big(\frac{t^2}{2} + \frac{4}{3} C' t^{3/2}\Big)$$

which gives the desired result. $\qquad\square$

Proof of Theorem 7.1. — Chose t'_1 such that $C(t'^2_1 + 2C't'^{3/2}_1) = \theta/2$ and set $t_1 = \max(1, t'_1)$. As a consequence, at time $t = t_1$ one has $v \leqslant \theta/2$ and

$$\mu u(t, x) \leqslant 2e^{-\mu t_1} \frac{1}{\sqrt{4\pi t_1}}\, a + \frac{\theta}{2} \leqslant \frac{2\theta}{3},$$

if $a < a_0$ for some a_0. Then the maximum principle yields that from this time $\mu u, v$ will always stay below the constant solution $2\theta/3$ of (3). And so, μu and v will tend to 0. $\qquad\square$

7.2. **Best case scenario:** $a = +\infty$. — In this subsection we take $\mu u_0 \equiv 1$. Since both the initial data and Equation (4) enjoy here a translation invariance in the x direction, u and v do not depend on x. We prove the following

Theorem 7.2. — *There exist $\mu^\pm > 0$ such that:*

   (a) *If $\mu > \mu^+$, μu and v converge uniformly to $1/(\mu(L + 1/\mu))$ as $t \to +\infty$.*
   (b) *If $\mu < \mu^-$, μu and v converge uniformly to 1 as $t \to +\infty$.*

Proof of (a). — Using Lemma 7.1 one gets, for $t \leqslant 1$: $v(t, x, y) \leqslant C\sqrt{t}$ (for some constant C different than the C in the afore mentioned Lemma). Using this in the equation for u, one gets

$$\mu u(t, x) \leqslant e^{-\mu t} + C\mu \int_0^t e^{-\mu(t-s)} \sqrt{s}\, \mathrm{d}s \leqslant e^{-\mu t} + C\mu t^{3/2}.$$

So, at $t_\mu = (\theta/(2C)/\mu)^{2/3}$, provided μ is large enough so that $e^{-\mu t_\mu} \leqslant \theta/2$ and $v \leqslant \theta$, i.e.,

$$\mu \geqslant \max\Big((2C/\theta)^2 \,|\ln(\theta/2)|^3, \frac{1}{2}\, (C/\theta)^2 \Big) =: \mu^+,$$

one has $\mu u, v \leqslant \theta$. By the comparison principle, this will hold for all $t > t_\mu$ and v never gets above θ anywhere. As a consequence, $\mu u(t), v(t, y)$ converge to a common limit $l \leqslant \theta$ satisfying $(L + 1/\mu)l = 1/\mu$ (conservation of mass). $\qquad\square$

Proof of (b). — The idea of the proof is simple: we investigate whether the sole diffusion is able to transfer enough mass from u to v so that in finite time v is above θ on a large enough interval $(-L_0, 0)$. The quantity L_0 is linked to Kanel' and Aronson-Weinberger [18, 2]. Using $v \geqslant 0$ and the strong parabolic maximum principle one gets $\mu u \geqslant e^{-\mu t}$. So that setting $\theta' = (1 + \theta)/2$ and $t_M = \frac{1}{\mu} \ln(1/\theta')$ one has $\mu u \geqslant \theta'$ while $t \leqslant t_M$, so that, by the maximum principle, Hopf's lemma and the positivity of f, up to time t_M we have $v \geqslant \underline{v}$ the solution of

$$
\begin{array}{c}
\hline
d\partial_y \underline{v} + \underline{v} = \theta' \\[4pt]
\partial_t \underline{v} - d\partial_{yy}^2 \underline{v} - d\varepsilon^2 \partial_{xx}^2 \underline{v} = 0 \\[4pt]
\partial_y \underline{v} = 0 \\
\hline
\end{array}
\tag{47}
$$

starting from $\underline{v}_0 = 0$. Observe that $\underline{v}$ is independent of x, so we will call it $\underline{v}(t, y)$ from now on. The function $w = \theta' - \underline{v}$ is easily decomposed as

$$
w(t, y) = \sum_{k \geqslant 0} e^{-\lambda_k t} \widetilde{w}_k(0) \cos\left(\sqrt{\lambda_k/d}\,(y + L)\right),
$$

where the $\lambda_k/d > 0$ solve $\sqrt{x} = \cotan(\sqrt{x}\,L)$ and $\sum \widetilde{w}_k(0) \cos\left(\sqrt{\lambda_k/d}(y + L)\right) = \theta'$. By the maximum principle we have, for $y \in (-L_0, 0)$:

$$
w \leqslant \theta' e^{-\lambda_1 t} \frac{\cos\left(\sqrt{\lambda_1/d}\,(L - L_0)\right)}{\cos\left(\sqrt{\lambda_1/d}\,L\right)} =: K e^{-\lambda_1 t},
$$

so that for $y \in (-L_0, 0)$, $v(t_M, y) \geqslant \theta' - K e^{-\lambda_1 t_M} \geqslant (1 + 3\theta)/4$ provided

$$
\mu \leqslant \lambda_1 \frac{\ln(1/\theta')}{\ln\left(4K/(1 - \theta)\right)} =: \mu^-
\tag{48}
$$

Chose L_0 large enough in the beginning so that an initial condition $\mu u(t_M), v(t_M, y) \geqslant (1 + 3\theta)/4$, for all $y \in (-L_0, 0)$, leads to invasion: $\mu u, v \to 1$ as $t \to \infty$. The existence of such an L_0 follows from Kanel', Aronson-Weinberger [18, 2] on $\mathbb{R}$. In our context, it is in fact simpler since total mass is confined in $(-L, 0)$ and a single point whereas in [18, 2] it can be spread on all $\mathbb{R}$. □

7.3. **Large $a < +\infty$.** — We use the best case scenario described above to prove the existence of large but finite a that will lead to invasion. Our proof relies on the fact that $\mathbf{1}_{(-a,a)}$ and $\mathbf{1}_{(-\infty,\infty)}$ are close in L^∞ weighted by some $\rho(x)$ with tails $e^{-|x|}$ and that such a weight preserves the semi-linear parabolic and monotone structure of the system (4). In particular, the "weighted equation" will have a locally (in time) Lipschitz continuous flow. Going back to the original solutions, this Lipschitz continuity becomes a uniform continuity on every compact subset.

Lemma 7.3. — *There exists a smooth weight $\rho(x) > 0$ such that $\rho(x) = e^{-|x|}$ for $|x| > 1$ and such that the following holds. Denote $\|(f, g)\|_X = \max\left(\|\rho f\|_{L^\infty(\mathbb{R})}, \|\rho g\|_{L^\infty(\Omega_L)}\right)$.*

For every $T, M > 0$, there is $C_{T,M} > 0$ independent of D such that

$$\sup_{\substack{0 \leqslant t \leqslant T \\ x \in (-M, M)}} (|u - \widetilde{u}| + |v - \widetilde{v}|) \leqslant C_{T,M} \|(u_0 - \widetilde{u}_0, v_0 - \widetilde{v}_0)\|_X$$

for every (u, v) and $(\widetilde{u}, \widetilde{v})$ solutions of (4) starting from respectively (u_0, v_0) and $(\widetilde{u}_0, \widetilde{v}_0)$.

REMARK 7.1. — Observe that the above Lemma could be stated for any $\rho_\alpha(x) = e^{-\alpha|x|}$ (with $\alpha > 0$) by changing the constants: this is due to the scaling invariance $(t, x, y) \to (\Lambda t, \sqrt{\Lambda}x, \sqrt{\Lambda}y)$ of Equation (4); indeed, ρ_α becomes ρ_1 in the rescaling by $\Lambda = \alpha^2$.

End of the proof of Theorem 2.7. — Once Lemma 7.3 is proved, the end of Theorem 2.7 follows easily. Indeed, let $u_0 = \mathbf{1}_{(-a,a)}$ and $v_0 = 0$ as well as $\widetilde{u}_0 = \mathbf{1}_{(-\infty,+\infty)}$ and $\widetilde{v}_0 = 0$. Observe that if $a > 1$, we have

$$\|(u_0 - \widetilde{u}_0, v_0 - \widetilde{v}_0)\|_X = e^{-a}.$$

Moreover, since $\mu < \mu^-$, by Theorem 7.2 above, there exists $T > 0$ such that

$$\mu\widetilde{u}(T, x), \widetilde{v}(T, x, y) \geqslant 1 - \delta/2,$$

with δ as in Theorem 4.1. By choosing

$$a > \max\{1, -\ln(\delta/(2C_{T,M}))\} =: a_1$$

(which does not depend on D), and applying Lemma 7.3 on $[0, T] \times [-M, M]$ (with M as in Theorem 4.1), one has

$$|\mu u(T, x) - \mu\widetilde{u}(T, x)| + |v(T, x, y) - \widetilde{v}(T, x, y)| \leqslant \delta/2 \quad \text{for all } -M < x < M.$$

As a consequence, $\mu u(T, x), v(T, x, y) \geqslant 1 - \delta$ for all $-M < x < M$. $\qquad\square$

Proof of Lemma 7.3. — The proof relies only on the parabolic maximum principle applied to a weighted equation. Let $\rho(x)$ define a positive $\mathcal{C}^2$ function such that $\rho(x) = e^{-|x|}$ for $|x| \geqslant 1$. Let (u, v) solve system (4). Observe that $(\mathfrak{u}, \mathfrak{v}) := (\rho u, \rho v)$ satisfies

$$\frac{\partial_t \mathfrak{u} + 2(\rho'/\rho)\partial_x \mathfrak{u} - \partial_{xx}^2 u = \mathfrak{v} - \left(\mu + (\rho''/\rho) - 2(\rho'/\rho)^2\right)\mathfrak{u}}{d\partial_y \mathfrak{v} = \mu\mathfrak{u} - \mathfrak{v}}$$

$$(49) \qquad \partial_t \mathfrak{v} + 2(d\rho'/D\rho)\partial_x \mathfrak{v} - (d/D)\partial_{xx}^2 \mathfrak{v} - d\partial_{yy}^2 \mathfrak{v}$$
$$= \rho f(\mathfrak{v}/\rho) - (d/D)\left((\rho''/\rho) - 2(\rho'/\rho)^2\right)\mathfrak{v}$$

$$\overline{\partial_y \mathfrak{v} = 0}$$

Equation (49) is a semilinear parabolic system, and thanks to the definition of ρ, has bounded coefficients. Moreover, the non-linearity $g(\mathfrak{v}) := \rho f(\mathfrak{v}/\rho)$ is Lipschitz with Lipschitz constant $\mathrm{Lip}\, f$. Let

$$C := \mathrm{Lip}\, f - \inf_{\mathbb{R}}\left((\rho''/\rho) - 2(\rho'/\rho)^2\right) > \mathrm{Lip}\, f - \frac{d}{D}\inf_{\mathbb{R}}\left((\rho''/\rho) - 2(\rho'/\rho)^2\right)$$

Now define similarly $(\widetilde{\mathfrak{u}}, \widetilde{\mathfrak{v}}) = (\rho\widetilde{u}, \rho\widetilde{v})$ and let

$$\mathfrak{U} := e^{-Ct}\left(\mathfrak{u} - \widetilde{\mathfrak{u}}\right), \quad \mathfrak{V} := e^{-Ct}\left(\mathfrak{v} - \widetilde{\mathfrak{v}}\right).$$

Observe that $(\mathfrak{U}, \mathfrak{V})$ satisfies

$$\partial_t\mathfrak{U} + 2(\rho'/\rho)\partial_x\mathfrak{U} - \partial_{xx}^2\mathfrak{U} = \mathfrak{V} - \mu\mathfrak{U} - \left((\rho''/\rho) - 2\left(\rho'/\rho\right)^2 + C\right)\mathfrak{U}$$
$$d\partial_y\mathfrak{V} = \mu\mathfrak{U} - \mathfrak{V}$$

$$(50) \qquad \begin{aligned} \partial_t\mathfrak{V} + 2(d\rho'/D\rho)\partial_x\mathfrak{V} - (d/D)\partial_{xx}^2\mathfrak{V} - d\partial_{yy}^2\mathfrak{V} \\ + \left((d/D)\left((\rho''/\rho) - 2(\rho'/\rho)^2\right) + \frac{g(\mathfrak{v}) - g(\widetilde{\mathfrak{v}})}{\mathfrak{v} - \widetilde{\mathfrak{v}}} + C\right)\mathfrak{V} = 0 \end{aligned}$$
$$\partial_y\mathfrak{V} = 0$$

By choice of C, the 0-order terms in parentheses in Equation (50) are positive, thus Equation (50) enjoys the maximum principle and the maximum and minimum values of $(\mu\mathfrak{U}, \mathfrak{V})$ are reached at initial time. Indeed, as usual if the maximum is reached by $\mathfrak{V}$, then either it is reached at initial time or it has to be reached on $y = 0$ but there Hopf's lemma gives the contradiction $\mu\mathfrak{U} > \mathfrak{V}$. It $\mathfrak{U}$ reaches it, a contradiction is obtained at this point by seeing that the left-hand side in the equation satisfied by $\mathfrak{U}$ is non-negative: thus $\mathfrak{V} > \mu\mathfrak{U}$. In the end we have for all $0 \leqslant t < T$:

$$\max(|(\mathfrak{u} - \widetilde{\mathfrak{u}})(t)|_{L^\infty(\mathbb{R})}, |(\mathfrak{v} - \widetilde{\mathfrak{v}})(t)|_{L^\infty(\Omega_L)})$$
$$\leqslant e^{CT}\max\left(|(\mathfrak{u} - \widetilde{\mathfrak{u}})(0)|_{L^\infty(\mathbb{R})}, |(\mathfrak{v} - \widetilde{\mathfrak{v}})(0)|_{L^\infty(\Omega_L)}\right),$$

i.e., for all $t < T$, $x \in \mathbb{R}$, $y \in [-L, 0]$:

$$\rho(x)\left(|u - \widetilde{u}|(t, x) + |v - \widetilde{v}|(t, x, y)\right) \leqslant 2e^{CT}\|(u_0 - \widetilde{u}_0, v_0 - \widetilde{v}_0)\|_X$$

and Lemma 7.3 follows by taking $C_{T,M} = 2e^{CT}\sup_{x \in (-M,M)} 1/\rho(x)$, which is $2e^{CT}e^M$ when M is large. Observe that $C_{T,M}$ depends only on T, M and Lip f. $\qquad \square$

Appendix: the heat kernel

We compute here the large time asymptotics of the solution $(u(t, x), v(t, x, y))$ to

$$(51) \qquad \begin{aligned} u_t - u_{xx} + \mu u - v(t, x, 0) &= 0 & (t > 0, \ x \in \mathbb{R}) \\ v_t - \varepsilon^2 v_{xx} - v_{yy} &= 0 & (t > 0, \ x \in \mathbb{R}, \ y \in (-L, 0)) \\ v_y(t, x, 0) &= \mu u(t, x) - v(t, x, 0) & (t > 0, \ x \in \mathbb{R}) \end{aligned}$$

with initial datum

$$(52) \qquad\qquad (u(0, x), v(0, x, y)) = (u_0(x), 0).$$

Notice that we have, without loss of generality, set $d = 1$. We limit ourselves to $u(t, x)$, as this is The quantity that will be useful to us. the method that we use is quite standard, a computationally much more involved case being treated in [10], where the diffusion on the road is represented by the fractional Laplacian.

PROPOSITION A.1. — *Set $a = (1 + \mu\varepsilon^2)/(1 + \mu)$. There are two constants $C > 0$ and $\omega > 0$, and a function $\delta(t)$ tending to 0 as $t \to +\infty$ such that we have, for $t \geqslant 1$*

$$\left| u(t, x) - (1 + \delta(t)) \int_{\mathbb{R}} \frac{e^{-(x-x')^2/4at}}{\sqrt{2\pi at}} u_0(x') \,\mathrm{d}x' \right| \leqslant C e^{-\omega t} \|u_0\|_{L^1(\mathbb{R})}.$$

Proof. — Let $(\widehat{u}(t, \xi), \widehat{v}(t, \xi, y))$ be the Fourier transform in x of (u, v); we have

(53)
$$\begin{aligned}
\widehat{u}_t + \xi^2 \widehat{u} + \mu \widehat{u} - \widehat{v}(t, x, 0) &= 0 && (t > 0, \ \xi \in \mathbb{R}) \\
\widehat{v}_t - \varepsilon^2 \widehat{v} - \widehat{v}_{yy} &= 0 && (t > 0, \ \xi \in \mathbb{R}, \ y \in (-L, 0)) \\
\widehat{v}_y(t, \xi, 0) &= \mu u(t, \xi) - v(t, \xi, 0) && (t > 0, \ \xi \in \mathbb{R})
\end{aligned}$$

with initial datum $(\widehat{u}(0, \xi), \widehat{v}(0, \xi, y)) = (\widehat{u}_0(x), 0)$.

1. *The case of large $|\xi|$.* — Let $A(\xi)$ be the operator acting on $C((-L, 0))$ with domain the set of all functions $w(y) \in C^2((-L, 0))$ such that $w_y(-L) = w_y(0) + w(0) = 0$, and defined by $A(\xi)w = -w_{yy} + \varepsilon^2 \xi^2 w$. Its first eigenvalue is (as is given by a simple computation) $\lambda_0 + \varepsilon^2 \xi^2$, with λ_0 being the first positive root of

$$\sqrt{\lambda L} \tan \sqrt{\lambda L} = 1;$$

an eigenfunction is $\cos(\sqrt{\lambda_0 L}\,(y + L))$, that we may bound from below by a real number δ_0. Thus the solution $w(t, \xi, y)$ of

$$w_t - w_{yy} + \varepsilon^2 \xi^2 w = 0, \ (t > 0, -L < y < 0), \quad w_y(t, \xi, -L) = w_y(t, \xi, 0) + w(t, \xi, 0) = 0,$$

with initial datum $w_0(\xi, y)$ satisfies

(54)
$$|w(t, \xi, y)| \leqslant \delta_0^{-1} \|w_0(\xi, .)\|_{L^\infty((-L, 0))}.$$

Now, multiply the equation for $\widehat{u}$ by $\widehat{u}/|\widehat{u}|$, this yields

$$\partial_t |\widehat{u}| + (\xi^2 + \mu)|\widehat{u}| \leqslant \mu |\widehat{v}(t, \xi, 0)|.$$

From (54) and the Duhamel formula we obtain

$$\partial_t |\widehat{u}| + (\xi^2 + \mu)|\widehat{u}| \leqslant \mu \delta_0^{-1} \int_0^t e^{-\lambda_0(t-s)} |\widehat{u}(s, \xi)| \,\mathrm{d}s.$$

Choose any $\omega_0 \in (0, \lambda_0)$, we will prove the inequality

(55)
$$|\widehat{u}(t, \xi)| \leqslant \frac{C|\widehat{u}_0(\xi)| e^{-\omega_0 t}}{\xi^4 - A} + |\widehat{u}_0(\xi)| e^{-(\xi^2 + \mu)t},$$

for a universal C, a suitably chosen $A > 0$ and $|\xi| \geqslant 10 A^{1/4}$. Set

$$U(t, \xi) = e^{\omega_0 t}(|\widehat{u}(t, \xi)| - |\widehat{u}_0(\xi)| e^{-(\xi^2 + \mu)t}),$$

we have

(56) $\quad \partial_t U + (\xi^2 + \mu - \omega_0)U \leqslant \mu \delta_0^{-1} \int_0^t e^{-\lambda_0(t-s)} U(s, \xi) \,\mathrm{d}s + \dfrac{\mu_0 \delta^{-1} |\widehat{u}_0(\xi)| e^{-(\lambda_0 - \omega_0)t}}{\xi^2 - (\lambda_0 - \omega_0)},$

The function $t \mapsto U(t, \xi)$ is either decreasing or has positive maxima. In the first case, because $U(0, \xi) = 0$ we have $|\widehat{u}(t, \xi)| \leqslant |\widehat{u}_0(\xi)| e^{-(\xi^2 + \mu)t}$, so (55) is proved. Let us

assume the contrary, and let t_0 such that $U(t,\xi) \leqslant U(t_0,\xi)$ for all $t \leqslant t_0$. In such a case we have, from (56):

$$\left(\xi^2 + \mu - \frac{\mu\delta_0^{-1}}{\lambda_0 - \omega_0}\right)U(t_0,\xi) \leqslant \frac{\mu_0\delta^{-1}|\widehat{u}_0(\xi)|e^{-(\lambda_0 - \omega_0)t_0}}{\xi^2 - (\lambda_0 - \omega_0)}.$$

So, (55) is once again proved.

2. *The case of intermediate $|\xi|$.* — We consider the range $\varepsilon_0 \leqslant |\xi| \leqslant A$, A being chosen so that (55) holds. Let us this time consider the operator $L(\xi)$, acting on $\mathbb{C} \times C((-L,0))$, its domain being all couples (u,v) in $\mathbb{C} \times C^2((-L,0))$ such that $w_y(-L) = 0$ and $w_y(0) + w(0) = \mu u$, and its action being given by

(57) $$L(\xi)(u,w) = (\xi^2 + \mu)u - w(0), -w_{ww} + \varepsilon^2\xi^2 w).$$

The family $L(\xi)$ is a family if sectorial operators with uniformly bounded coefficients. Moreover, by Krein-Rutman's theorem, for $|\xi| \in \varepsilon_0, A)$, and $\varepsilon \in (0,1)$, $L(\xi)$ has a bottom eigenvalue $\lambda_0(\xi)$. Possibly, it depends on ε, but with a common positive lower bound depending on ε_0 that we call ω_0. So, there is $\theta_0 \in (0\pi/2)$ such that the path $\gamma = \omega_0 + \mathbb{R}e^{i\theta_0}$ encloses $\sigma(L(\xi))$, for all ξ in the range that we consider. We have

(58) $$e^{-tL(\xi)}(\widehat{u}_0,\widehat{v}_0) = \frac{1}{2i\pi}\int_\gamma e^{-\lambda t}(\lambda I - L(\xi))^{-1}(\widehat{u}_0,0)\,d\lambda,$$

an expression that admits a bound of the form

(59) $$\|e^{-tL(\xi)}(\widehat{u}_0(\xi),0)\|_{L^\infty((-L,0))} \leqslant Ce^{-\omega_0 t}|\widehat{u}_0(\xi)|.$$

3. *The case of small $|\xi|$.* — Fix $\varepsilon_0 > 0$ small so that the following (finite set of) considerations are true. Here we simply perform a Laplace transform of (53) and we still call $\widehat{u}(\lambda,\xi)$ the unknown in the new variables (λ,ξ). This leads (after some standard algebra) to the system

(60) $$\left(\xi^2 - \lambda - \frac{\mu\sqrt{\lambda - \varepsilon^2\xi^2}\tan(L\sqrt{\lambda - \varepsilon^2\xi^2})}{1 - \sqrt{\lambda - \varepsilon^2\xi^2}\tan(L\sqrt{\lambda - \varepsilon^2\xi^2})}\right)\widehat{u}(\lambda,\xi) = \widehat{u}_0(\xi).$$

The factor before $\widehat{u}(\lambda,\xi)$ vanishes, for $|\xi| \leqslant \varepsilon_0$, at a unique $\lambda(\xi)$ satisfying:

(61) $$\lambda(\xi) \sim \frac{(1+\mu\varepsilon^2)\xi^2}{1+\mu},$$

the next zeroes being further away, uniformly in ξ, by the principle of isolated zeroes. Then, (58) remains valid and a standard inverse Laplace transform computation yields the existence of a function $\delta(t) = o_{t\to+\infty}(1)$ such that

(62) $$\widehat{u}(t,\xi) - (1+\delta(t))e^{-t\lambda(\xi)}\widehat{u}_0 = O(e^{-\omega_0 t})|\widehat{u}_0|.$$

The proposition is proved by taking the inverse Fourier transform of $\widehat{u}$, and putting together estimates (55), (59), (61) and (62). □

References

[1] D. G. Aronson – "Bounds for the fundamental solution of a parabolic equation", *Bull. Amer. Math. Soc.* **73** (1967), p. 890–896.

[2] D. G. Aronson & H. F. Weinberger – "Multidimensional nonlinear diffusion arising in population genetics", *Adv. in Math.* **30** (1978), no. 1, p. 33–76.

[3] B. Audoly, H. Berestycki & Y. Pomeau – "Réaction diffusion en écoulement stationnaire rapide", *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre* **328** (2000), no. 3, p. 255–262.

[4] H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre & L. Rossi – "The effect of a line with nonlocal diffusion on Fisher-KPP propagation", *Math. Models Methods Appl. Sci.* **25** (2015), no. 13, p. 2519–2562.

[5] H. Berestycki & L. Nirenberg – "Travelling fronts in cylinders", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9** (1992), no. 5, p. 497–572.

[6] H. Berestycki, J.-M. Roquejoffre & L. Rossi – "The influence of a line with fast diffusion on Fisher-KPP propagation", *J. Math. Biol.* **66** (2013), no. 4-5, p. 743–766.

[7] P. Constantin, A. Kiselev, A. Oberman & L. Ryzhik – "Bulk burning rate in passive-reactive diffusion", *Arch. Rational Mech. Anal.* **154** (2000), no. 1, p. 53–91.

[8] P. Constantin, A. Kiselev & L. Ryzhik – "Quenching of flames by fluid advection", *Comm. Pure Appl. Math.* **54** (2001), no. 11, p. 1320–1342.

[9] P. Constantin, J.-M. Roquejoffre, L. Ryzhik & N. Vladimirova – "Propagation and quenching in a reactive Burgers-Boussinesq system", *Nonlinearity* **21** (2008), no. 2, p. 221–271.

[10] A.-C. Coulon Chalmin – "Fast propagation in reaction-diffusion equations with fractional diffusion", PhD Thesis, Université de Toulouse, 2014.

[11] L. Dietrich – "Existence of travelling waves for a reaction-diffusion system with a line of fast diffusion", *Appl. Math. Res. Express. AMRX* (2015), no. 2, p. 204–252.

[12] _____ , "Velocity enhancement of reaction-diffusion fronts by a line of fast diffusion", *Trans. Amer. Math. Soc.* (2016), online.

[13] Y. Du & H. Matano – "Convergence and sharp thresholds for propagation in nonlinear diffusion problems", *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 2, p. 279–312.

[14] A. Fannjiang, A. Kiselev & L. Ryzhik – "Quenching of reaction by cellular flows", *Geom. Funct. Anal.* **16** (2006), no. 1, p. 40–69.

[15] P. C. Fife & J. B. McLeod – "The approach of solutions of nonlinear diffusion equations to travelling front solutions", *Arch. Rational Mech. Anal.* **65** (1977), no. 4, p. 335–361.

[16] D. Gilbarg & N. S. Trudinger – *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[17] F. Hamel & A. Zlatoš – "Speed-up of combustion fronts in shear flows", *Math. Ann.* **356** (2013), no. 3, p. 845–867.

[18] Ja. I. Kanel' – "Stabilization of the solutions of the equations of combustion theory with finite initial functions", *Mat. Sb. (N.S.)* **65 (107)** (1964), p. 398–413.

[19] A. Kiselev & A. Zlatoš – "Quenching of combustion by shear flows", *Duke Math. J.* **132** (2006), no. 1, p. 49–72.

[20] M. A. Lewis, S. V. Petrovskii & J. R. Potts – *The mathematics behind biological invasions*, Interdisciplinary Applied Mathematics, vol. 44, Springer, 2016.

[21] A. Mellet, J. Nolen, J.-M. Roquejoffre & L. Ryzhik – "Stability of generalized transition fronts", *Comm. Partial Differential Equations* **34** (2009), no. 4-6, p. 521–552.

[22] A. Novikov & L. Ryzhik – "Boundary layers and KPP fronts in a cellular flow", *Arch. Rational Mech. Anal.* **184** (2007), no. 1, p. 23–48.

[23] L. Roques, J.-P. Rossi, H. Berestycki, J. Rousselet, J. Garnier, J.-M. Roquejoffre, L. Rossi, S. Soubeyrand & C. Robinet – "Modeling the spatio-temporal dynamics of the pine processionary moth", in *Processionary Moths and Climate Change: An Update* (A. Roques, ed.), Springer Netherlands, 2015, p. 227–263.

[24] *Vespa velutina* Lepeletier, 1836, Inventaire National du Patrimoine Naturel, `http://inpn.mnhn.fr/espece/cd_nom/433589`.

[25] N. Vladimirova, P. Constantin, A. Kiselev, O. Ruchayskiy & L. Ryzhik – "Flame enhancement and quenching in fluid flows", *Combust. Theory Model.* **7** (2003), no. 3, p. 487–508.

[26] A. Zlatoš – "Sharp transition between extinction and propagation of reaction", *J. Amer. Math. Soc.* **19** (2006), no. 1, p. 251–263.

[27] _____, "Reaction-diffusion front speed enhancement by flows", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), no. 5, p. 711–726.

Laurent Dietrich, Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA 15213, U.S.A.
E-mail : `ldi@cmu.edu`
Url : `http://www.math.cmu.edu/~ldietric/`

Jean-Michel Roquejoffre, Institut de Mathématiques de Toulouse, Université Paul Sabatier
118 route de Narbonne, F-31062 Toulouse Cedex 4, France
E-mail : `jean-michel.roquejoffre@math.univ-toulouse.fr`