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GALOIS IRREDUCIBILITY IMPLIES COHOMOLOGY FREENESS FOR KHT SHIMURA VARIETIES

by Pascal Boyer

Abstract. — Given a KHT Shimura variety with an action of its unramified Hecke algebra $\mathbb{T}$, we proved in [7], see also [12] for other PEL Shimura varieties, that its localized cohomology groups at a generic maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, happen to be free. In this work, we obtain the same result for $\mathfrak{m}$ such that its associated Galois $\overline{\mathbb{F}}_\ell$-representation $\overline{\rho_{\mathfrak{m}}}$ is irreducible, under the hypothesis that $[F(\exp(2i\pi/\ell):F]>d$, where F is the reflex field, d the dimension of the KHT Shimura variety and ℓ the residual characteristic.

Résumé (L'irréductibilité galoisienne implique la liberté cohomologique pour les variétés de Shimura de type KHT)

Étant donnée une variété de Shimura unitaire de type KHT de dimension relative $d-1$ sur son corps reflex F et munie de l'action de son algèbre de Hecke $\mathbb{T}$ non ramifiée, nous prouvons dans [7], voir aussi [12] pour les autres variétés de Shimura de type PEL, que ses groupes de $\overline{\mathbb{Z}}_\ell$-cohomologie localisés en un idéal maximal générique $\mathfrak{m}$ de $\mathbb{T}$, sont libres. Dans ce travail, sous l'hypothèse que $[F(\exp(2i\pi/\ell):F]>d$, nous montrons le même résultat pour $\mathfrak{m}$ tel que sa $\overline{\mathbb{F}}_\ell$-représentation galoisienne associée, $\overline{\rho_{\mathfrak{m}}}$, est irréductible.

Contents

Introduction

From Matsushima's formula and computations of $(\mathfrak{G},K_\infty)$-cohomology, we know that tempered automorphic representations contributions in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. If one

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considers cohomology with coefficients in a very regular local system, then only tempered representations can contribute so that all of the cohomology is concentrated in middle degree.

For $\overline{\mathbb{Z}}_\ell$-coefficients and Shimura varieties of Kottwitz-Harris-Taylor types, we proved in [9], whatever the weight of the coefficients is, when the level is large enough at ℓ, there are always nontrivial torsion cohomology classes, so that the $\overline{\mathbb{F}}_\ell$-cohomology can not be concentrated in middle degree. Thus if one wants an $\overline{\mathbb{F}}_\ell$-analog of the previous $\overline{\mathbb{Q}}_\ell$-statement, one must cut off some part of the cohomology.

In [7] for KHT Shimura varieties, and more generally in [12] for any PEL proper Shimura variety, we obtain such a result under some genericity hypothesis which can be stated as follows. Let $(\mathrm{Sh}_K)_{K\subset G(\mathbb{A}^\infty)}$ be a tower, indexed by open compact subgroups K of $G(\mathbb{A}^\infty)$, of compact Shimura varieties of Kottwitz type associated to some similitude group G and of relative dimension $d-1$ over its reflex field $F=EF^+$ where F^+ is totally real and $E/\mathbb{Q}$ is an imaginary quadratic extension. Let then $\mathfrak{m}$ be a system of Hecke eigenvalues appearing in $H^{n_0}(\mathrm{Sh}_K\times_F\overline{F},\overline{\mathbb{F}}_\ell)$. By the main result of [20], one can attach to such an $\mathfrak{m}$, a mod ℓ Galois representation

$$\overline{\rho_{\mathfrak{m}}}:\mathrm{Gal}(\overline{F}/F)\longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_\ell).$$

From [12, Def. 19], we say that $\mathfrak{m}$ is generic (resp. decomposed generic) at some split p in E, if for all places v of F dividing p, the set $\{\lambda_1,\dots,\lambda_d\}$ of eigenvalues of $\overline{\rho_{\mathfrak{m}}}(\mathrm{Frob}_v)$ satisfies $\lambda_i/\lambda_j\notin\{q_v^{\pm1}\}$ for all $i\neq j$ (resp. and are pairwise distinct), where q_v is the cardinal of the residue field at v. Then under the hypothesis(1) that there exists such a p with $\mathfrak{m}$ generic at p, the integer n_0 above is necessarily equal to the relative dimension of Sh_K. In particular the $H^i(\mathrm{Sh}_K\times_F\overline{F},\overline{\mathbb{Z}}_\ell)_{\mathfrak{m}}$ are all torsion-free.

In this work we consider the particular case of Kottwitz-Harris-Taylor Shimura varieties Sh_K of [16] associated to inner forms of GL_d. Exploiting the fact, which is particular to these Shimura varieties, that the non supersingular Newton strata are geometrically induced, we are then able to prove the following result which happens to be useful at least for our approach of Ihara's lemma, cf. [10].

Theorem. — *We suppose that $[F(\exp(2i\pi/\ell):F]>d$. Let $\mathfrak{m}$ be a system of Hecke eigenvalues such that $\overline{\rho_{\mathfrak{m}}}$ is irreducible, then the localized cohomology groups of* Sh_K *with coefficients in any $\overline{\mathbb{Z}}_\ell$-local system V_ξ, are all free.*

Remarks

– From [4, §4], we know that outside the middle degree the irreducible Galois subquotients of $H^i(\mathrm{Sh}_K\times_F\overline{F},\overline{\mathbb{Z}}_\ell)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell$ are of dimension strictly less than d, so that over $\overline{\mathbb{Q}}_\ell$, the cohomology, localized at $\mathfrak{m}$, is concentrated in middle degree. The previous theorem then tells us it is the same for the $\overline{\mathbb{F}}_\ell$-cohomology.

– Note that Koshikawa, cf. [18], starting from [7] and using techniques from group theory, proved a similar result in low dimension.

(1) In their new preprint, Caraiani and Scholze explained that, from an observation of Koshikawa, one can replace decomposed generic by simply generic, in their main statement.

Lemma. — *There exist infinitely many places v of F such that q_v, the order of the residue field of F at v, is of order strictly greater than d in $(\mathbb{Z}/\ell\mathbb{Z})^\times$.*

Proof. — The hypothesis $[F(\exp(2i\pi/\ell) : F] > d$ is equivalent to the fact that the minimal polynomial of $\exp(2i\pi/\ell)$ over F is of degree $\delta > d$. By Cebotarev's theorem, there exists then a set of places v of strictly positive density such that the minimal polynomial of $\exp(2i\pi/\ell)$ over the residue field at v is also of degree δ. Recall that the roots of this minimal polynomial are $\exp(2i\pi/\ell)^{q_v^k}$ with $k = 0, \dots, \delta - 1$ with $\exp(2i\pi/\ell)^{q_v^\delta} = 1$ in the residue field. We then deduce that the order of q_v modulo ℓ divides δ but, as $\exp(2i\pi/\ell)^{q_v^k} \neq 1$ in the residue field for $0 \leqslant k < \delta$, then q_v is of order $\delta > d$ modulo ℓ. □

This property is used at three places in the proof.

– For a place v as above, there is no irreducible cuspidal representation π_v of $\mathrm{GL}_g(F_v)$ with $g > 1$ such that its reduction modulo ℓ has a supercuspidal support made of characters, cf. the remark after Notation 1.1.4. This simplification is completely harmless and if one wants to take care of these cuspidal representations, it suffices to use [11, Prop. 2.4.1].

– With this hypothesis we also note that the pro-order of $\mathrm{GL}_d(\mathcal{O}_v)$ is invertible modulo ℓ so that, concerning torsion cohomology classes, we can easily pass from infinite to maximal level at v, cf. for example Lemma 3.1.15.

– Finally in the last section, arguing by contradiction, we are able to construct a sequence of intervals $\{\lambda, \lambda q_v, \dots, \lambda q_v^r\}$ contained in the set of eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ so that at the end we obtain the full set $\{\lambda q_v^n \in \overline{\mathbb{F}}_\ell \mid n \in \mathbb{Z}\}$ which is of order the order of q_v modulo ℓ, which is trivially absurd if this order is strictly greater than the dimension d of $\overline{\rho}_{\mathfrak{m}}$.

The proof takes place in four main steps.

(1) First, in Section 3.1, we analyze the torsion in the cohomology of Harris-Taylor perverse sheaves at some place v and with infinite level at v, and we focus on ℓ-torsion cohomology classes with maximal non degeneracy level, cf. Lemma 3.1.17.

(2) Recall, cf. [6], that one can define an exhaustive filtration of stratification $\mathrm{Fill}^\bullet(\Psi_v)$ of the nearby cycles perverse sheaf whose graded parts are perverse sheaves

$$^{p}j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v)) \lhook\joinrel\twoheadrightarrow_+ P(t, \pi_v) \lhook\joinrel\twoheadrightarrow_+ {}^{p+}j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v)),$$

cf. Section 2.2 for more details about p and $p+$ perverse structures and bimorphisms. The main point is that the constructions of [6] is of geometric nature and so works whatever the coefficients are.

Then on can compute the cohomology of the Shimura variety through the spectral sequence associated to the filtration $\mathrm{Fill}^\bullet(\Psi_v)$, whose E_1 terms are given by the cohomology groups of the Harris-Taylor perverse $\overline{\mathbb{Z}}_\ell$-sheaves. The main point, cf. Lemma 3.2.5, is that the ℓ-torsion of the cohomology of the Shimura variety with infinite level at v does not have an irreducible subquotient whose cuspidal support is made of characters.

(3) We then consider levels which are of Iwahori type at v and infinite at a place w verifying the same hypothesis as v above. We can then resume the previous result for w, i.e., the cohomology of the Shimura variety with infinite level at w does not have an irreducible subquotient whose cuspidal support is made of characters. We then focus on the various lattices of $H^{d-1}_{\mathrm{free}}(\mathrm{Sh}_K \times_F \overline{F}, V_{\xi,\overline{\mathbb{Q}}_\ell})_{\widetilde{\mathfrak{m}}}$ where $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ can be seen as a near equivalence class $\Pi_{\widetilde{\mathfrak{m}}}$ in the sense of of [22]. The ones given by the $\overline{\mathbb{Z}}_\ell$-cohomology, are only slightly modified from the ones given by the cohomology of the Harris-Taylor perverse sheaves, in the sense that the ℓ-torsion of the cokernel measuring the difference between two such lattices, as a representation of $\mathrm{GL}_d(F_w)$, does not have any irreducible generic subquotient with cuspidal support made of characters, cf. Proposition 3.3.17. Moreover if the torsion sub-module of $H^{d-1}(\mathrm{Sh}_K \times_F \overline{F}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ were not trivial, we prove, using the geometrical induced structure of the Newton strata, that it would exist $\widetilde{\mathfrak{m}}$ such that the lattices mentioned above, are not isomorphic, cf. Proposition 3.3.18.

(4) Finally in Section 3.4, for any $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, $H^{d-1}(\mathrm{Sh}_K \times_F \overline{F}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ induces a quotient stable lattice $\Gamma_{\widetilde{\mathfrak{m}}}$ of $(\Pi^{\infty}_{\widetilde{\mathfrak{m}}})^K \otimes \rho_{\widetilde{\mathfrak{m}}}$. As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, then this lattice is isomorphic to a tensor product of a stable lattice of $(\Pi^{\infty}_{\widetilde{\mathfrak{m}}})^K$ by a stable lattice of $\rho_{\widetilde{\mathfrak{m}}}$. Then the idea is to start from the filtration of the free quotient of $H^{d-1}(\mathrm{Sh}_K \times_F \overline{F}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ given by the filtration of the nearby cycles perverse sheaf, so that, using repeatedly diagrams as (3.4.2), we arrive at $\Gamma_{\widetilde{\mathfrak{m}}}$. In the process we are able to construct an increasing sequence of intervals contained in the set of eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}\, v)$ so that at the end we obtain a full set $\{\lambda q_v^n \mid n \in \mathbb{Z}\}$ which is of order the order of q_v modulo ℓ which is, by hypothesis, strictly greater than the dimension of $\overline{\rho}_{\mathfrak{m}}$, which is absurd.

We refer the reader to the introduction of Section 3 for more details.

1. Reminder from [7]

1.1. Representations of $\mathrm{GL}_d(K)$. — Let $K/\mathbb{Q}_p$ be a finite extension with $\mathcal{O}_K$ its ring of integers, ϖ an uniformizer, and κ its residue field with order q. We denote by $|\bullet|$ its absolute value. For a representation π of $\mathrm{GL}_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \operatorname{val} \circ \det}.$$

The Zelevinsky line associated to π is by definition the set $\{\pi\{n\} \mid n \in \mathbb{Z}\}$.

Notations 1.1.1. — For π_1 and π_2 representations of respectively $\mathrm{GL}_{n_1}(K)$ and $\mathrm{GL}_{n_2}(K)$, we will denote by

$$\pi_1 \times \pi_2 := \operatorname{ind}_{P_{n_1,n_1+n_2}(K)}^{\mathrm{GL}_{n_1+n_2}(K)} \pi_1\{n_2/2\} \otimes \pi_2\{-n_1/2\},$$

the normalized parabolic induced representation where for any sequence

$$\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d),$$

we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$\mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2 - r_1} \times \cdots \times \mathrm{GL}_{r_k - r_{k-1}}.$$

Recall that a representation ϱ of $\mathrm{GL}_d(K)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_\ell$.

Definition 1.1.2 (see [24, §9] and [4, §1.4]). — Let g be a divisor of $d=sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_g(K)$. The induced representation

$$\pi\{(1-s)/2\}\times\pi\{(3-s)/2\}\times\cdots\times\pi\{(s-1)/2\}$$

admits a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi)$ (resp. $\mathrm{Speh}_s(\pi)$); it is a generalized Steinberg (resp. Speh) representation.

Moreover the induced representation $\mathrm{St}_t(\pi\{-r/2\})\times\mathrm{Speh}_r(\pi\{t/2\})$ (resp. of $\mathrm{St}_{t-1}(\pi\{(-r-1)/2\})\times\mathrm{Speh}_{r+1}(\pi\{(t-1)/2\}))$ has a unique irreducible subspace (resp. quotient), denoted $LT_\pi(t-1,r)$.

Remark. — These representations $LT_\pi(t-1,r)$ appear in the cohomology of the Lubin-Tate spaces, cf. [3].

Proposition 1.1.3 (cf. [23, III.5.10]). — *Let π be an irreducible cuspidal representation of $\mathrm{GL}_g(K)$ with a stable $\overline{\mathbb{Z}}$-lattice*(2)*, then its reduction modulo ℓ is irreducible and cuspidal but not necessary supercuspidal.*

The supercuspidal support of the reduction modulo ℓ of a cuspidal representation, is a segment associated to some irreducible $\overline{\mathbb{F}}_\ell$-supercuspidal representation ϱ of $\mathrm{GL}_{g_{-1}(\varrho)}(F_v)$ with $g=g_{-1}(\varrho)t$, where t is either equal to 1 or of the following shape $t=m(\varrho)\ell^u$ with $u\geqslant 0$ and where $m(\varrho)$ is defined as follows.

Notation 1.1.4. — We denote by $m(\varrho)$ the order of the Zelevinsky line $\{\varrho(\delta)\mid\delta\in\mathbb{Z}\}$ of ϱ if it is not equal to 1, otherwise $m(\varrho)=\ell$.

Remark. — When ϱ is the trivial representation then $m(1_v)$ is either the order of q modulo ℓ when it is >1, otherwise $m(1_v)=\ell$. We say that such a π_v is of ϱ-type u with $u\geqslant -1$.

Notation 1.1.5. — For ϱ an irreducible $\overline{\mathbb{F}}_\ell$-supercuspidal representation, we denote by Cusp_ϱ (resp. $\mathrm{Cusp}_\varrho(u)$ for some $u\geqslant -1$) the set of equivalence classes of irreducible $\overline{\mathbb{Q}}_\ell$-cuspidal representations whose reduction modulo ℓ has for supercuspidal support a segment associated to ϱ (resp. of ϱ-type u).

Let $u\geqslant 0$, $\pi_{v,u}\in\mathrm{Cusp}_\varrho(u)$ and $\tau=\pi_{v,u}[s]_D$. Let then denote by ι the image of $\mathrm{Speh}_s(\varrho)$ by the Jacquet-Langlands correspondence modulo ℓ defined in [13, §1.2.4]. Then the reduction modulo ℓ of τ is isomorphic to

$$\iota\{-(m(\tau)-1)/2\}\oplus\iota\{-(m(\tau)-3)/2\}\oplus\cdots\oplus\iota\{(m(\tau)-1)/2\} \tag{1.1.6}$$

where $\iota\{n\}:=\iota\otimes q^{-n\,\mathrm{val}\circ\mathrm{nrd}}$.

(2) We say that π is integral.

We now recall the notion of level of non degeneracy from [1, §4]. The mirabolic subgroup $M_d(K)$ of $\mathrm{GL}_d(K)$ is the sub-group of matrices with last row $(0,\dots,0,1)$: we denote by

$$V_d(K)=\{(m_{i,j}\in P_d(K)\mid m_{i,j}=\delta_{i,j} \text{ for } j<n\}.$$

its unipotent radical. We fix a nontrivial character ψ of K and let θ be the character of $V_d(K)$ defined by $\theta((m_{i,j}))=\psi(m_{d-1,d})$. For $G=\mathrm{GL}_r(K)$ or $M_r(K)$, we denote by $\mathrm{Alg}(G)$ the abelian category of algebraic representations of G and, following [1], we introduce

$$\Psi^-:\mathrm{Alg}(M_d(K))\longrightarrow \mathrm{Alg}(\mathrm{GL}_{d-1}(K),\qquad \Phi^-:\mathrm{Alg}(M_d)\longrightarrow\mathrm{Alg}(M_{d-1}(K))$$

defined by $\Psi^-=r_{V_d,1}$ (resp. $\Phi^-=r_{V_d,\theta}$) the functor of V_{d-1} coinvariants (resp. (V_{d-1},θ)-coinvariants), cf. [1, 1.8]. We also introduce the normalized compact induced functor

$$\Psi^+:=i_{V,1}:\mathrm{Alg}(\mathrm{GL}_{d-1}(K))\longrightarrow\mathrm{Alg}(M_d(K)),$$
$$\Phi^+:=i_{V,\theta}:\mathrm{Alg}(M_{d-1}(K))\longrightarrow\mathrm{Alg}(M_d(K)).$$

Proposition 1.1.7 ([1, p. 451])
- *The functors Ψ^-, Ψ^+, Φ^- and Φ^+ are exact.*
- $\Phi^-\circ\Psi^+=\Psi^-\circ\Phi^+=0$.
- *Ψ^- (resp. Φ^+) is left adjoint to Ψ^+ (resp. Φ^-) and the following adjunction maps*

$$\mathrm{Id}\longrightarrow\Phi^-\Phi^+,\qquad \Psi^+\Psi^-\longrightarrow\mathrm{Id},$$

are isomorphisms, with an exact sequence

$$0\longrightarrow\Phi^+\Phi^-\longrightarrow\mathrm{Id}\longrightarrow\Psi^+\Psi^-\longrightarrow 0.$$

Definition 1.1.8. — For $\tau\in\mathrm{Alg}(M_d(K))$, the representation

$$\tau^{(k)}:=\Psi^-\circ(\Phi^-)^{k-1}(\tau)$$

is called the k-th derivative of τ. If $\tau^{(k)}\neq 0$ and $\tau^{(m)}=0$ for all $m>k$, then $\tau^{(k)}$ is called the highest derivative of τ.

Notation 1.1.9 (cf. [24, 4.3]). — Let $\pi\in\mathrm{Alg}(\mathrm{GL}_d(K))$ (or $\pi\in\mathrm{Alg}(M_d(K))$. The maximal number k such that $(\pi_{|M_d(K)})^{(k)}\neq(0)$ is called the level of non-degeneracy of π and denoted by $\lambda(\pi)$. We can also iterate the construction so that at the end we obtain a partition $\underline{\lambda(\pi)}$ of d.

Definition 1.1.10. — A representation π of $\mathrm{GL}_d(K)$, over $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$, is then said generic if its level of non degeneracy $\lambda(\pi)$ is equal to d.

Remark. — Over $\overline{\mathbb{Q}}_\ell$, an irreducible generic representation of $\mathrm{GL}_d(K)$ looks like $\mathrm{St}_{t_1}(\pi_1)\times\cdots\times\mathrm{St}_{t_r}(\pi_r)$, where $\pi_1,\dots,\pi_r$ are irreducible cuspidal representations. Note moreover that the reduction modulo ℓ of any irreducible generic representation admits a unique generic irreducible subquotient.

In the following we will be interested in representations $\mathrm{St}_{t_1}(\chi_1)\times\cdots\times\mathrm{St}_{t_r}(\chi_r)$ where $\chi_1,\ldots,\chi_r$ are unramified characters.

Notation 1.1.11. — Associated to a partition $\underline{d}=(d_1\geqslant d_2\geqslant d_s)$ of $d=\sum_{i=1}^s d_i$, we consider the following Iwahori type subgroup of $\mathrm{GL}_d(\mathcal{O}_K)$:

$$\mathrm{Iw}_v(\underline{d}):=\big\{g\in\mathrm{GL}_d(\mathcal{O}_K)\mid (g \bmod \varpi)\in P_{d_1<d_1+d_2<\cdots<d}(\kappa)\big\}.$$

Recall the well-known following lemma.

Lemma 1.1.12. — *With the previous notations,* $\mathrm{St}_{t_1}(\chi_1)\times\cdots\times\mathrm{St}_{t_r}(\chi_r)$ *has nontrivial invariant vectors under* $\mathrm{Iw}(\underline{d})$ *if, and only if,* $\underline{d}$ *is smaller, for the Bruhat order, to the dual partition associated to* $(t_1,\ldots,t_r)$.

Remark. — Recall that one way to obtain the dual partition is to use Fejer's diagrams. To a partition $(d_1\geqslant d_2\geqslant d_r)$ one can associate a Fejer diagram with rows of respective size $d_1,\ldots,d_r$. Then one can read this Fejer diagram through its columns whose size gives the dual partition associated to $(d_1\geqslant\cdots\geqslant d_r)$.

Examples

– If $t_1=\cdots=t_r=1$ with $r=d$, then the dual partition of $(1,\ldots,1)$ is (d) and $\chi_1\times\cdots\times\chi_r$ has nontrivial invariant vectors under $\mathrm{Iw}(d)=\mathrm{GL}_d(\mathcal{O}_K)$ and so under all the Iwahori type subgroup $\mathrm{Iw}(\underline{d})$.

– At the opposite $\mathrm{St}_d(\chi)$ has nontrivial invariant vectors under $\mathrm{Iw}(1,\ldots,1)$ and it is the only Iwahori type subgroup with this property.

– $\mathrm{Speh}_d(\chi)$ has nontrivial invariant vectors under $\mathrm{GL}_d(\mathcal{O}_K)$.

– $LT_\chi(t-1,d-t)$ which can be seen as a subspace of $\mathrm{St}_t(\chi_v\{(t-d)/2\}\times\mathrm{Speh}_{d-t}(\chi_v\{t/2)$ has nontrivial invariant vectors under $\mathrm{Iw}(\underline{d})$, where $\underline{d}$ is the dual partition of $(t,1,\ldots,1)$, i.e., $\underline{d}=(d-t+1,1,\ldots,1)$. Moreover it has no nontrivial invariant vectors under $\mathrm{Iw}(\underline{d}')$ for any $\underline{d}'$ strictly greater than $\underline{d}$.

1.2. Shimura varieties of KHT type. — Let $F=F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ is totally real with a fixed real embedding $\tau:F^+\hookrightarrow\mathbb{R}$. For a place v of F, we will denote by

– F_v the completion of F at v,
– $\mathcal{O}_v$ the ring of integers of F_v,
– ϖ_v a uniformizer,
– q_v the cardinal of the residue field $\kappa(v)=\mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F, of dimension d^2 such that at every place x of F, either B_x is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta\in B^{*=-1}$, denote by $\sharp_\beta$ the involution $x\mapsto x^{\sharp_\beta}=\beta x^*\beta^{-1}$ and let $G/\mathbb{Q}$ be the group of similitudes, denoted by G_τ in [16], defined for every $\mathbb{Q}$-algebra R by

$$G(R)\simeq\{(\lambda,g)\in R^\times\times(B^{\mathrm{op}}\otimes_\mathbb{Q}R)^\times\mid gg^{\sharp_\beta}=\lambda\}$$

with $B^{\mathrm{op}} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$, split $x = yy^c$ in E, then

$$G(\mathbb{Q}_x) \simeq (B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{\mathrm{op}})^\times, \tag{1.2.1}$$

where, identifying places of F^+ over x with places of F over y, $x = \prod_i z_i$ in F^+.

Convention. — For a place $x = yy^c$ of $\mathbb{Q}$ split in E and z a place of F over y, we shall make throughout the text the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{\mathrm{op}})^\times$ in the formula (1.2.1).

In [16], the authors justify the existence of some G like above such that moreover

– if x is a place of $\mathbb{Q}$ non split in E then $G(\mathbb{Q}_x)$ is quasi split;
– the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [16, bottom of p. 90], a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said to be *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

Notation 1.2.2. — Denote by $\mathcal{I}$ the set of open compact subgroups small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $\mathrm{Sh}_{I,\eta} \to \mathrm{Spec}\, F$ for the associated Shimura variety of Kottwitz-Harris-Taylor type.

Definition 1.2.3. — Denote by Spl the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq \ell$ is split in E and let $B_v^\times \simeq \mathrm{GL}_d(F_v)$. For each $I \in \mathcal{I}$, we write $\mathrm{Spl}(I)$ for the subset of Spl of places which does not divide I.

In the sequel, v and w will denote places of F in Spl. For such a place v, the scheme $\mathrm{Sh}_{I,\eta}$ has a projective model $\mathrm{Sh}_{I,v}$ over $\mathrm{Spec}\, \mathcal{O}_v$ with special fiber Sh_{I,s_v}. For I going through $\mathcal{I}$, the projective system $(\mathrm{Sh}_{I,v})_{I \in \mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that any w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}^{-1}$ and $\mathrm{Art}^{-1} : W_v^{\mathrm{ab}} \simeq F_v^\times$ is the isomorphism of Artin sending the geometric Frobenius to a uniformizer.

Notations 1.2.4. — For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ is denoted by

$$\mathrm{Sh}_{I,\bar{s}_v} =: \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant 1} \supset \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant 2} \supset \cdots \supset \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant d},$$

where $\mathrm{Sh}_{I,\bar{s}_v}^{=h} := \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h} - \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h+1}$ is an affine scheme, smooth of pure dimension $d-h$ built up by the geometric points whose connected part of their Barsotti-Tate group is of rank h. For each $1 \leqslant h < d$, write

$$i_h : \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h} \hookrightarrow \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant 1}, \quad j^{\geqslant h} : \mathrm{Sh}_{I,\bar{s}_v}^{=h} \hookrightarrow \mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h},$$

and $j^{=h} = i_h \circ j^{\geqslant h}$.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_\ell$ be a fixed embedding and write Φ for the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ whose restriction to E equals σ_0. There exists then, cf. [16, p. 97], an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_\ell$ and $(d+1)$-uples $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$, where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$. We then denote by

$$V_{\xi,\overline{\mathbb{Z}}_\ell}$$

the associated $\overline{\mathbb{Z}}_\ell$-local system on $\mathrm{Sh}_{\mathcal{I}}$. Recall that an irreducible automorphic representation Π is said ξ-cohomological if there exists an integer i such that

$$H^i\big((\mathrm{Lie}\ \ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$. Let $d_\xi^i(\Pi_\infty)$ be the dimension of this cohomology group.

1.3. Cohomology of the Newton strata

Notation 1.3.1. — For $1 \leqslant h \leqslant d$, let $\mathcal{I}_v(h)$ be the set of open compact subgroups

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where

$$K_v(m_1) = \mathrm{Ker}\big(\mathrm{GL}_{d-h}(\mathcal{O}_v) \longrightarrow \mathrm{GL}_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1}))\big).$$

We then denote by $[H^i(h,\xi)]$ (resp. $[H^i_!(h,\xi)]$) the image of

$$\varinjlim_{I\in\mathcal{I}_v(h)} H^i(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v,1}, V_{\xi,\overline{\mathbb{Q}}_\ell}[d-h]), \qquad \text{resp. } \varinjlim_{I\in\mathcal{I}_v(h)} H^i(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v,1}, j^{\geqslant h}_{1,!} V_{\xi,\overline{\mathbb{Q}}_\ell}[d-h])$$

inside the Grothendieck $\mathrm{Groth}(v,h)$ of admissible representations of

$$G(\mathbb{A}^\infty) \times \mathrm{GL}_{d-h}(F_v) \times \mathbb{Z}.$$

Remark. — An element $\sigma \in W_v$ acts through $-\deg\sigma \in \mathbb{Z}$ and $\Pi_{p_v,0}(\mathrm{Art}^{-1}(\sigma))$. We moreover consider the action of $\mathrm{GL}_h(F_v)$ through $\mathrm{val}\circ\det : \mathrm{GL}_h(F_v) \to \mathbb{Z}$ and finally $P_{h,d}(F_v)$ through its Levi factor $\mathrm{GL}_h(F_v) \times \mathrm{GL}_{d-h}(F_v)$, i.e., its unipotent radical acts trivially.

From [7, Prop. 3.6], for any irreducible tempered automorphic representation Π of $G(\mathbb{A})$ and for every $i \neq 0$, the $\Pi^{\infty,v}$-isotypic component of $[H^i(h,\xi)]$ and $[H^i_!(h,\xi)]$ are zero. About the case $i=0$, for Π an irreducible automorphic tempered ξ-cohomological representation, its local component at v is generic and so looks like

$$\Pi_v \simeq \mathrm{St}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{St}_{t_u}(\pi_{v,u}),$$

where for $i = 1, \dots, u$, $\pi_{v,i}$ is an irreducible cuspidal representation of $\mathrm{GL}_{g_i}(F_v)$.

Proposition 1.3.2 (cf. [7, Prop. 3.9]). — *With the previous notations, we order the representations $\pi_{v,i}$ such that the first r ones are unramified characters. Then the $\Pi^{\infty,v}$-isotypic component of $[H^0(h,\xi)]$ is equals to*

$$\Big(\frac{\sharp\,\mathrm{Ker}^1(\mathbb{Q},G)}{d}\sum_{\Pi'\in\mathcal{U}_G(\Pi^{\infty,v})} m(\Pi')d_\xi(\Pi'_\infty)\Big)\Big(\sum_{1\leqslant k\leqslant r:\ t_k=h} \Pi_v^{(k)} \otimes \chi_{v,k}\chi^{(d-h)/2}\Big),$$

where

– $\mathrm{Ker}^1(\mathbb{Q},G)$ *is the subset of elements of* $H^1(\mathbb{Q},G)$ *which become trivial in* $H^1(\mathbb{Q}_{p'},G)$ *for every prime* p'*;*

– $\Pi_v^{(k)} := \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_{k-1}}(\chi_{v,k-1}) \times \mathrm{St}_{t_{k+1}}(\chi_{v,k+1}) \times \cdots \times \mathrm{St}_{t_u}(\chi_{v,u})$ *and*

– $\Xi : \frac{1}{2}\mathbb{Z} \to \overline{\mathbb{Z}}_\ell^\times$ *is defined by* $\Xi(1/2) = q_v^{1/2}$*.*

– $\mathcal{U}_G(\Pi^{\infty,v})$ *is the set of equivalence classes of irreducible automorphic representations* Π' *of* $G(\mathbb{A})$ *such that* $(\Pi')^{\infty,v} \simeq \Pi^{\infty,v}$*.*

Remark. — With the notations of [7], we only consider Harris-Taylor perverse sheaves associated to the trivial representation which then selects the unramified characters to describe its cohomology groups.

Note also that if $[H^0(h,\xi)]$ has nontrivial invariant vectors under some open compact subgroup $I \in \mathcal{I}_v(h)$ which is maximal at v, then the local component of Π at v is of the following shape $\mathrm{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \chi_{v,d-h}$, where the $\chi_{v,i}$ are unramified characters.

Definition 1.3.3. — For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}^S_{\mathrm{abs}} := \prod_{x\notin S} \mathbb{T}_{x,\mathrm{abs}}$ the abstract unramified Hecke algebra, where $\mathbb{T}_{x,\mathrm{abs}} \simeq \overline{\mathbb{Z}}_\ell[X^{\mathrm{un}}(T_x)]^{W_x}$ for T_x a split torus, W_x is the spherical Weyl group and $X^{\mathrm{un}}(T_x)$ is the set of $\overline{\mathbb{Z}}_\ell$-unramified characters of T_x.

Example. — For $w \in \mathrm{Spl}$, we have

$$\mathbb{T}_{w,\mathrm{abs}} = \overline{\mathbb{Z}}_\ell\big[T_{w,i} :\ i = 1,\dots,d\big],$$

where $T_{w,i}$ is the characteristic function of

$$\mathrm{GL}_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w,\dots,\varpi_w}^{i},\overbrace{1,\dots,1}^{d-i})\,\mathrm{GL}_d(\mathcal{O}_w) \subset \mathrm{GL}_d(F_w).$$

Notation 1.3.4. — Let $\mathbb{T}^S_\xi$ be the image of $\mathbb{T}^S_{\mathrm{abs}}$ inside

$$\bigoplus_{i=0}^{2d-2} \varinjlim_{I} H^i(\mathrm{Sh}_{I,\overline{\eta}}, V_{\xi,\overline{\mathbb{Q}}_\ell}),$$

where the limit is taken over the ideals I which are maximal at each place outside S. For an open compact subgroup I maximal at each place outside S, we will also denote by $\mathbb{T}^S_{I,\xi}$ the image of $\mathbb{T}^S_{\mathrm{abs}}$ inside $H^{d-1}(\mathrm{Sh}_{I,\overline{\eta}}, V_{\xi,\overline{\mathbb{Q}}_\ell})$.

Let state some remarks about these Hecke algebras.

– The torsion cohomology classes give also sets of Satake parameters and we can ask if they correspond to maximal ideals of $\mathbb{T}^S_\xi$ which we choose to define through the $\overline{\mathbb{Q}}_\ell$-coefficients. In [7], we proved that the torsion cohomology classes of $H^i(\mathrm{Sh}_{I,\overline{\eta}}, V_{\xi,\overline{\mathbb{Z}}_\ell})$ with I maximal at each place outside S, can be lifted in characteristic zero. More precisely, for any place $v' \in \mathrm{Spl}$ not in S, there exists a maximal ideal $\mathfrak{m}$ of $(\mathbb{T}^{S\cup\{v'\}}_{I\,\mathrm{Iw}_{v'},\xi})_{\mathfrak{m}}$, such that the set of Satake parameters of our torsion cohomology class is associated to $\mathfrak{m}$, where $\mathrm{Iw}_{v'}$ is the Iwahori subgroup of $\mathrm{GL}_d(\mathcal{O}_{v'})$.

– As explained in the introduction, we will consider maximal ideals $\mathfrak{m}$ such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible, so that the $\overline{\mathbb{Q}}_\ell$-cohomology groups are all concentrated in middle degree, i.e., in degree 0 if we deal with perverse sheaves.

– With the notations of Section 2.2 about the Harris-Taylor local systems, in [4], we proved that, except for the $\mathrm{GL}_d(F_v)$-action, the irreducible subquotients of the $\overline{\mathbb{Q}}_\ell$-cohomology groups of $j_!^{\geqslant tg} HT(\pi_v, \Pi_t)$ or ${}^p j_{!*}^{\geqslant tg} HT(\pi_v, \Pi_t)$ are also subquotients of the cohomology of $\mathrm{Sh}_{I,\overline{\eta}}$.

The minimal prime ideals of $\mathbb{T}_\xi^S$ are the prime ideals above the zero ideal of $\overline{\mathbb{Z}}_\ell$ and are then in bijection with the prime ideals of $\mathbb{T}_\xi^S \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$. To such an ideal, which corresponds to giving a collection of Satake parameters, is then associated a unique near equivalence class in the sense of [22], denoted by $\Pi_{\widetilde{\mathfrak{m}}}$, which is the finite set of irreducible automorphic cohomological representations whose multi-set of Satake parameters at each place $x \in \mathrm{Unr}(I)$, is given by the multi-set $S_{\widetilde{\mathfrak{m}}}(x)$ of roots of the Hecke polynomial

$$P_{\widetilde{\mathfrak{m}},w}(X) := \sum_{i=0}^{d} (-1)^i q_w^{i(i-1)/2}\, T_{w,i,\widetilde{\mathfrak{m}}} X^{d-i} \in \overline{\mathbb{Q}}_\ell[X],$$

i.e.,

$$S_{\widetilde{\mathfrak{m}}}(w) := \bigl\{\lambda \in \mathbb{T}_\xi^S \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell/\widetilde{\mathfrak{m}} \simeq \overline{\mathbb{Q}}_\ell \mid P_{\widetilde{\mathfrak{m}},w}(\lambda) = 0\bigr\}.$$

Thanks to [16] and [22], we denote by

$$\rho_{\widetilde{\mathfrak{m}}} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell)$$

the Galois representation associated to any $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$. Recall that the reduction modulo ℓ of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of $\mathfrak{m}$, and was denoted above by $\overline{\rho_{\mathfrak{m}}}$. For every $w \in \mathrm{Spl}(I)$, we also denote by $S_{\mathfrak{m}}(w)$ the multi-set of Satake parameters modulo ℓ at w given as the multi-set of roots of

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^{d} (-1)^i q_w^{i(i-1)/2}\, \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_\ell[X],$$

i.e.,

$$S_{\mathfrak{m}}(w) := \bigl\{\lambda \in \mathbb{T}_\xi^S/\mathfrak{m} \simeq \overline{\mathbb{F}}_\ell \mid P_{\mathfrak{m},w}(\lambda) = 0\bigr\}.$$

2. About the nearby cycles perverse sheaf

Our strategy to compute the cohomology of the KHT-Shimura variety $\mathrm{Sh}_{I,\overline{\eta}}$ with coefficients in $V_{\xi,\overline{\mathbb{Z}}_\ell}$, is to realize it as the outcome of the nearby cycles spectral sequence at some place $v \in \mathrm{Spl}$.

Note that the role of the local system $V_{\xi,\overline{\mathbb{Z}}_\ell}$ associated to ξ is completely harmless when dealing with sheaves: one just has to add a tensor product with it to all the statements without the index ξ. In the following we will sometimes not mention the index ξ in the statements to make formulas more readable. Of course, when looking at the cohomology groups, the role of $V_{\xi,\overline{\mathbb{Z}}_\ell}$ is crucial as it selects the automorphic representations which contribute to the cohomology.

2.1. The case where the level at v is maximal. — By the smooth base change theorem, we have $H^i(\mathrm{Sh}_{I,\overline{\eta}_v}, V_\xi) \simeq H^i(\mathrm{Sh}_{I,\overline{s}_v}, V_\xi)$. As for each h such that $1 \leqslant h \leqslant d-1$, the open Newton stratum $\mathrm{Sh}^{=h}_{I,\overline{s}_v}$ is affine, then $H^i_c(\mathrm{Sh}^{=h}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h])$ is zero for $i<0$ and free for $i=0$. Using this property and the following short exact sequence of free perverse sheaves

$$0 \longrightarrow i_{h+1\to h,*} V_{\xi,\overline{\mathbb{Z}}_\ell,|\,\mathrm{Sh}^{\geqslant h+1}_{I,\overline{s}_v}}[d-h-1] \longrightarrow j^{\geqslant h}_! j^{\geqslant h,*} V_{\xi,\overline{\mathbb{Z}}_\ell,|\,\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}}[d-h] \longrightarrow V_{\xi,\overline{\mathbb{Z}}_\ell,|\,\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}}[d-h] \longrightarrow 0,$$

where $i_{h+1\to h} : \mathrm{Sh}^{\geqslant h+1}_{I,\overline{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}$, we then obtain for every $i>0$ an exact sequence

$$(2.1.1)\quad 0 \longrightarrow H^{-i-1}(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h]) \longrightarrow H^{-i}(\mathrm{Sh}^{\geqslant h+1}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h-1]) \longrightarrow 0,$$

and for $i=0$, a long exact sequence

$$(2.1.2)\quad 0 \longrightarrow H^{-1}(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h]) \longrightarrow H^{0}(\mathrm{Sh}^{\geqslant h+1}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h-1]) \longrightarrow H^0(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, j^{\geqslant h}_! j^{\geqslant h,*} V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h]) \longrightarrow H^0(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h]) \longrightarrow \cdots$$

In [7], arguing by induction from $h=d$ to $h=1$, we prove that for a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S_\xi$ such that $S_\mathfrak{m}(v)$ does not contain any subset of the form $\{\alpha, q_v\alpha\}$, all the cohomology groups $H^i(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_\mathfrak{m}$ are free: note that in order to deal with $i \geqslant 0$, one has to use Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$(2.1.3)\qquad H^0_{\mathrm{free}}(\mathrm{Sh}^{\geqslant h+1}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h-1])_\mathfrak{m} \longrightarrow H^0(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, j^{\geqslant h}_! j^{\geqslant h,*} V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h])_\mathfrak{m}.$$

In particular it lifts in characteristic zero to some free subquotient of

$$H^0(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}, j^{\geqslant h}_! j^{\geqslant h,*} V_{\xi,\overline{\mathbb{Z}}_\ell}[d-h])_\mathfrak{m}.$$

Main assumption. — We argue by contradiction and we suppose there exists a finite level I maximal at v such that there exist nontrivial torsion cohomology classes in the $\mathfrak{m}$-localized cohomology of $\mathrm{Sh}_{I,\overline{\eta}_v}$ with coefficients in $V_{\xi,\overline{\mathbb{Z}}_\ell}$. We then fix such a finite level I.

Proposition 2.1.4 (cf. [7, Lem. 4.13]). — *Consider $h_0(I)$ maximal such that there exists $i \in \mathbb{Z}$ with $H^{d-h_0(I)+i}(\mathrm{Sh}^{\geqslant h_0(I)}_{I,\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m},\mathrm{tor}} \neq (0)$. Then we have the following properties:*

- *$i = 0, 1$;*
- *for all $1 \leqslant h \leqslant h_0(I)$ and $i < h - h_0(I)$,*

$$H^{d-h+i}(\mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v}\, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m},\mathrm{tor}} = (0),$$

while for $i = h - h_0(I)$ it is nontrivial.

Remark. — Note that any system of Hecke eigenvalues $\mathfrak{m}$ of $\mathbb{T}^S_\xi$ inside the torsion of some $H^i(\mathrm{Sh}_{I,\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})$ lifts in characteristic zero, i.e., is associated to a minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}^{S\cup\{v\}}_\xi$: the level of $\widetilde{\mathfrak{m}}$ outside v can still be taken to be I^v, and, at v, using the remark following Proposition 1.3.2, $\pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_{h_0(I)+1}(\chi_v)\times\chi_{v,1}\times\cdots\times\chi_{v,d-h_0(I)-1}$, where $\chi_v,\chi_{v,1},\dots,\chi_{v,d-h_0(I)-1}$ are characters of $F_v^\times$.

2.2. Harris-Taylor perverse sheaves over $\overline{\mathbb{Z}}_\ell$. — Consider now the ideals $I^v(n) := I^vK_v(n)$, where $K_v(n) := \mathrm{Ker}(\mathrm{GL}_d(\mathcal{O}_v) \twoheadrightarrow \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n))$. Recall then that $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely this means there exists a closed subscheme $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d}(F_v)$ and such that

$$\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v} = \mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,\overline{1_h}} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n),$$

meaning that $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v}$ is the disjoint union of copies of $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,\overline{1_h}}$ indexed by $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ and exchanged by the action of $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)$.

Notation 2.2.1. — For any $g \in \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$, we denote by $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,g}$ the *pure* Newton stratum defined as the image of $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,\overline{1_h}}$ by g. Its closure in $\mathrm{Sh}_{I^v(n),\overline{s}_v}$ is then denoted by $\mathrm{Sh}^{\geqslant h}_{I^v(n),\overline{s}_v,g}$.

Let then denote by $\mathfrak{m}^v$ the multiset of Hecke eigenvalues given by $\mathfrak{m}$ but outside v and introduce for any representation Π_h of $\mathrm{GL}_h(F_v)$:

$$H^i(\mathrm{Sh}^{\geqslant h}_{I^v(\infty),\overline{s}_v,\overline{1_h}}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}^v}\otimes\Pi_h := \varinjlim_n H^i(\mathrm{Sh}^{\geqslant h}_{I^v(n),\overline{s}_v,\overline{1_h}}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}^v}\otimes\Pi_h,$$

as a representation of $\mathrm{GL}_h(F_v)\times\mathrm{GL}_{d-h}(F_v)$, where $g\in\mathrm{GL}_h(F_v)$ acts on Π_h as well as on $H^i(\mathrm{Sh}^{\geqslant h}_{I^v(n),\overline{s}_v,\overline{1_h}}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}^v}$ through the determinant map $\det:\mathrm{GL}_h(F_v)\twoheadrightarrow F_v^\times$. Note moreover that the unipotent radical of $P_{h,d}(F_v)$ acts trivially on these cohomology groups. We then introduce their induced version

$$H^i(\mathrm{Sh}^{\geqslant h}_{I^v(\infty),\overline{s}_v}, \Pi_h\otimes V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}^v} \simeq \mathrm{ind}_{P_{h,d}(F_v)}^{\mathrm{GL}_d(F_v)} H^i(\mathrm{Sh}^{\geqslant h}_{I^v(\infty),\overline{s}_v,\overline{1_h}}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}^v}\otimes\Pi_h.$$

More generally, with the notations of [3], replace now the trivial representation by an irreducible cuspidal representation π_v of $\mathrm{GL}_g(F_v)$ for some $1\leqslant g\leqslant d$.

Notations 2.2.2. — Let $1\leqslant t\leqslant s:=\lfloor d/g\rfloor$ and let Π_t be any representation of $\mathrm{GL}_{tg}(F_v)$. We then denote by

$$\widetilde{HT}_{1,\overline{\mathbb{Z}}_\ell}(\pi_v,\Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}},\overline{\mathbb{Z}}_\ell}\otimes\Pi_t\otimes\Xi^{(tg-d)/2}$$

the $\overline{\mathbb{Z}}_\ell$-Harris-Taylor local system on the Newton stratum $\mathrm{Sh}^{=tg}_{I,\overline{s}_v,\overline{1_{tg}}}$, where

– $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}},\overline{\mathbb{Z}}_\ell}$ is defined in [16, §IV-1], by means of Igusa varieties attached to the representation $\pi_v[t]_D$ of the division algebra of dimension $(tg)^2$ over F_v associated to $\mathrm{St}_t(\pi_v)$ by the Jacquet-Langlands correspondence,

– $\Xi:\frac{1}{2}\mathbb{Z}\to\overline{\mathbb{Z}}_\ell^\times$ defined by $\Xi(1/2)=q^{1/2}$.

We also introduce the induced version

$$\widetilde{HT}(\pi_v,\Pi_t)_{\overline{\mathbb{Z}}_\ell} := \Big(\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}},\overline{\mathbb{Z}}_\ell}\otimes\Pi_t\otimes\Xi^{(tg-d)/2}\Big)\times_{P_{tg,d}(F_v)}\mathrm{GL}_d(F_v),$$

where the unipotent radical of $P_{tg,d}(F_v)$ acts trivially and the action of

$$\Big(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{\rm et} \end{pmatrix}, \sigma_v\Big) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d}(F_v) \times W_v$$

is given by

– the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{(tg-d)/2}$,

– and the action of $(g^{\infty,v}, g_v^{\rm et}, \mathrm{val}(\det g_v^c) - \deg\sigma_v) \in G(\mathbb{A}^{\infty,v}) \times \mathrm{GL}_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_\ell}(\pi_v[t]_D)_{\overline{1_{tg}},\overline{\mathbb{Z}}_\ell} \otimes \Xi^{(tg-d)/2}$.

We also introduce

$$HT(\pi_v, \Pi_t)_{1,\overline{\mathbb{Z}}_\ell} := \widetilde{HT}(\pi_v, \Pi_t)_{1,\overline{\mathbb{Z}}_\ell}[d-tg],$$

and the perverse sheaf

$$P(t,\pi_v)_{1,\overline{\mathbb{Z}}_\ell} := {}^p j^{=tg}_{\overline{1_{tg}},!*} HT(\pi_v, \mathrm{St}_t(\pi_v))_{1,\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell}$ and $P(t,\pi_v)_{\overline{\mathbb{Z}}_\ell}$, where

$$j^{=h}_{\overline{1_h}} = i^h \circ j^{\geqslant h}_{\overline{1_h}} : \mathrm{Sh}^{=h}_{I,\overline{s}_v,\overline{1_h}} \hookrightarrow \mathrm{Sh}^{\geqslant h}_{I,\overline{s}_v} \hookrightarrow \mathrm{Sh}_{I,\overline{s}_v}$$

and $\mathbb{L}^\vee$, the dual of $\mathbb{L}$, is the local Langlands correspondence which sends geometric frobenii to uniformizers. Finally we will also use the index ξ in the notations, for example $HT_\xi(\pi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell}$, when we twist the sheaf with $V_{\xi,\overline{\mathbb{Z}}_\ell}$.

With the previous notations, from (1.1.6), we deduce the following equality in the Grothendieck group of Hecke-equivariant local systems

$$m(\varrho)\ell^u\Big[\mathbb{F}\mathcal{L}_{\xi,\overline{\mathbb{Z}}_\ell}(\pi_{v,u}[t]_D)\Big] = \Big[\mathbb{F}\mathcal{L}_{\xi,\overline{\mathbb{Z}}_\ell}(\pi_{v,-1}[tm(\varrho)\ell^u]_D)\Big]. \tag{2.2.3}$$

We now focus on the perverse Harris-Taylor sheaves. Note first, cf. [17, (2.2–2.4)], that over $\overline{\mathbb{Z}}_\ell$, there are two notions of intermediate extension associated to the two classical t-structures p and $p+$: essentially they come from the choice about $\mathbb{F}_\ell$ as a sheaves over the point, represented by the complex $\mathbb{Z}_\ell \xrightarrow{\times \ell} \mathbb{Z}_\ell$ and where one decides to put the zero grading on the second factor, which corresponds to the p-structure, or on the first one which gives the $p+$-structure. So for every $\pi_v \in \mathrm{Cusp}_\varrho$ of $\mathrm{GL}_g(F_v)$ and $1 \leqslant t \leqslant d/g$, we can define:

$$ {}^p j^{=tg}_{!*} HT(\pi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell} \hookrightarrow\!\!\!\!\rightarrow_+ {}^{p+} j^{=tg}_{!*} HT(\pi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell}, \tag{2.2.4}$$

the symbol $\hookrightarrow\!\!\!\!\rightarrow_+$ meaning bimorphism, i.e., both a monomorphism and an epimorphism, so that the cokernel for the t-structure p (resp. the kernel for $p+$) has support in $\mathrm{Sh}^{\geqslant tg+1}_{I,\overline{s}_v}$. When π_v is a character, i.e., when $g=1$, the associated bimorphisms are isomorphisms, as explained in the following lemma, but in general they are not.

Lemma 2.2.5. — *With the previous notations, we have an isomorphism*

$$ {}^p j^{\geqslant h}_{\overline{1_h},!*} HT(\chi_v, \Pi_h)_{1,\overline{\mathbb{Z}}_\ell} \simeq {}^{p+} j^{\geqslant h}_{\overline{1_h},!*} HT(\chi_v, \Pi_h)_{1,\overline{\mathbb{Z}}_\ell}.$$

Proof. — Recall that $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}$ is smooth over $\operatorname{Spec}\overline{\mathbb{F}}_p$. Up to a modification of the action of the fundamental group through the character χ_v, we have

$$HT(\chi_v,\Pi_h)_{1,\overline{\mathbb{Z}}_\ell}[h-d]=(\overline{\mathbb{Z}}_\ell)_{|\,\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}}\otimes\Pi_h.$$

Then $HT(\chi_v,\Pi_h)_{1,\overline{\mathbb{Z}}_\ell}$ is perverse for both t-structures, with

$$i^{h\leqslant +1,*}_{\overline{1_h}}HT(\chi_v,\Pi_h)_{1,\overline{\mathbb{Z}}_\ell}\in {}^p\mathcal{D}^{<0}\quad\text{and}\quad i^{h\leqslant +1,!}_{\overline{1_h}}HT(\chi_v,\Pi_h)_{1,\overline{\mathbb{Z}}_\ell}\in {}^{p+}\mathcal{D}^{\geqslant 1}.\qquad\square$$

Remark. — One of the main results of [11, Prop. 2.4.1] is the fact that the previous lemma holds for any $\pi_v\in\mathrm{Cusp}_\varrho(-1)$. As explained in the introduction, with the hypothesis on the order of q_v modulo ℓ, which is supposed to be strictly greater than d, for $\varrho=\mathbb{1}_v$ being the trivial character, we do not need to bother about the representations $\pi_v\in\mathrm{Cusp}_{\mathbb{1}_v}(u)$ for $u\geqslant 0$, cf. the remark after Notation 1.1.4.

2.3. Filtrations of the nearby cycles perverse sheaf. — Let us denote by[3]

$$\Psi_v:=R\Psi_{\eta_v}(\overline{\mathbb{Z}}_\ell[d-1])((d-1)/2)$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ of Sh_I. We also set $\Psi_{\xi,v}:=\Psi_v\otimes V_{\xi,\overline{\mathbb{Z}}_\ell}$. Let $\mathrm{Scusp}_{\overline{\mathbb{F}}_\ell}(g)$ be the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_\ell$-supercuspidal representations of $\mathrm{GL}_g(F_v)$. In [2, Prop. 3.3.4], we prove the following splitting:

$$\Psi_v\simeq\bigoplus_{g=1}^{d}\bigoplus_{\varrho\in\mathrm{Scusp}_{\overline{\mathbb{F}}_\ell}(g)}\Psi_\varrho,\tag{2.3.1}$$

with the property that the irreducible subquotients of

$$\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell\simeq\bigoplus_{\pi_v\in\mathrm{Cusp}_\varrho}\Psi_{\pi_v}$$

are exactly the perverse Harris-Taylor sheaves, of level I, associated to an irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of some $\mathrm{GL}_g(F_v)$ such that the supercuspidal support of the reduction modulo ℓ of π_v is a segment associated to the inertial class ϱ.

This splitting relies on the various filtrations defined over $\overline{\mathbb{Z}}_\ell$, of Ψ_v constructed by means of the Newton stratification, cf. [8]. Using the adjunction morphism $j^{=t}_!j^{=t,*}\to\mathrm{Id}$ as in [6], we then define a filtration of $\Psi_{\xi,\varrho}$

$$\mathrm{Fil}^0_!(\Psi_{\xi,\varrho})\lhook\joinrel\relbar\joinrel\rightarrow\mathrm{Fil}^1_!(\Psi_{\xi,\varrho})\lhook\joinrel\relbar\joinrel\rightarrow\mathrm{Fil}^2_!(\Psi_{\xi,\varrho})\lhook\joinrel\relbar\joinrel\rightarrow\cdots\lhook\joinrel\relbar\joinrel\rightarrow\mathrm{Fil}^d_!(\Psi_{\xi,\varrho}),$$

where the symbol $\lhook\joinrel\relbar\joinrel\rightarrow$ means a monomorphism such that the cokernel is torsion-free, which here means that $\mathrm{Fil}^t_!(\Psi_{\xi,\varrho})$ is the saturated image of $j^{=t}_!j^{=t,*}\Psi_{\xi,\varrho}\to\Psi_{\xi,\varrho}$. We then denote by $\mathrm{gr}^k_!(\Psi_{\xi,\varrho})$ the graded parts and we have a spectral sequence:

$$E^{p,q}_{!,\varrho,1}=H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v},\mathrm{gr}^{-p}_!(\Psi_{\xi,\varrho}))\implies H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v},\Psi_{\xi,\varrho}).\tag{2.3.2}$$

(3) We decide not to add I in the list of indexes.

Remark. — Over $\overline{\mathbb{Q}}_\ell$, in [6] we proved that $\mathrm{Fil}_!^k(\Psi_{\xi,\varrho}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ is $\operatorname{Ker} N^k$ where N is the monodromy operator at v.

We can refine this filtration to define a filtration $\mathrm{Fil}^\bullet(\mathrm{gr}_!^k(\Psi_{\xi,\varrho}))$ of $\mathrm{gr}_!^k(\Psi_{\xi,\varrho})$ where, over $\overline{\mathbb{Q}}_\ell$, $\mathrm{Fil}^\bullet(\mathrm{gr}_!^k(\Psi_{\xi,\pi_v}))$ coincides with the filtration by iterated images of N, i.e., $\mathrm{gr}^r(\mathrm{gr}_!^k(\Psi_{\xi,\pi_v})) = \operatorname{Im} N^r \cap \operatorname{Ker} N^k$, so that we recover the usual monodromy bi-filtration of [3].

We then obtain, cf. [6], an exhaustive filtration of stratification $\mathrm{Fill}^\bullet(\Psi_{\xi,\varrho})$ of $\Psi_{\xi,\varrho}$ whose graded parts are free, isomorphic to some free perverse Harris-Taylor sheaves. Let us denote by $\mathrm{grr}^k(\Psi_{\xi,\varrho}) := \mathrm{Fill}^k(\Psi_{\xi,\varrho})/\mathrm{Fill}^{k-1}(\Psi_{\xi,\varrho})$ the graded parts of this exhaustive filtration. We then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{grr}^{-p}(\Psi_{\xi,\varrho})) \Longrightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\xi,\varrho}), \tag{2.3.3}$$

where we recall the bimorphisms

$$\begin{aligned} {}^p j_{!*}^{=tg} HT_\xi(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{\mathbb{Z}}_\ell}((1-t+2i)/2) \lhook\joinrel\twoheadrightarrow_+ \mathrm{grr}^k(\Psi_{\xi,v}) \\ \lhook\joinrel\twoheadrightarrow_+ {}^{p+} j_{!*}^{=tg} HT_\xi(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{\mathbb{Z}}_\ell}((1-t+2i)/2), \end{aligned} \tag{2.3.4}$$

for some irreducible cuspidal representation π_v of $\mathrm{GL}_g(F_v)$ with $1 \leqslant t \leqslant d/g$ and $0 \leqslant i \leqslant \lfloor d/g \rfloor - 1$ and of type ϱ.

Remark. — In [11] we proved that, following the previous process, then all the previous graded parts of Ψ_ϱ are isomorphic to p-intermediate extensions. In the following we will only consider the case where ϱ is the trivial character so that as the order of q_v being supposed to be $> d$, then Cusp_ϱ is made of characters in which case, cf. Lemma 2.2.5, the p and $p+$ intermediate extensions coincide. Note that in the following we will not use the results of [11].

3. Irreducibility implies freeness

Recall, cf. the main assumption in Section 2.1, that we argue by contradiction assuming there exist nontrivial cohomology classes in some of the $H^i(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$. The strategy is then to choose a place $v \in \mathrm{Spl}_I$ such that the order of q_v modulo ℓ is strictly greater than d and to allow ramification at v, either infinitely with $I^v(\infty)$ as denoted in the next paragraph, or of Iwahori type.

In the arguments we need to consider another place $w \neq v$ verifying the same hypothesis as v, i.e., $w \in \mathrm{Spl}_I$ and such that the order of q_w modulo ℓ is strictly greater than d. We will also allow to increase infinitely the level at w. As the order of both q_v and q_w is supposed to be strictly greater than d, then the functors of invariants by any open compact subgroups either at v or w, are exact. In particular as there exist nontrivial torsion classes in level I in some degree i, when I is maximal at v and w, there also exist nontrivial torsion classes in degree i, whatever the level J

such that $J^{v,w}=I^{v,w}$ is. In particular when the level at v is infinite, from the splitting

$$\begin{aligned}H^i(\mathrm{Sh}_{I^v(\infty),\bar\eta_v},V_{\xi,\overline{\mathbb{Z}}_\ell}[d-1])_{\mathfrak{m}} &\simeq H^i(\mathrm{Sh}_{I^v(\infty),\bar s_v},\Psi_{\xi,v})_{\mathfrak{m}}\\ &\simeq \bigoplus_{g=1}^{d}\bigoplus_{\varrho\in\mathrm{Scusp}_{\overline{\mathbb{F}}_\ell}(g)} H^i(\mathrm{Sh}_{I^v(\infty),\bar s_v},\Psi_{\xi,\varrho})_{\mathfrak{m}},\end{aligned}$$

there also exist nontrivial torsion classes in $H^i(\mathrm{Sh}_{I^v(\infty),\bar s_v},\Psi_{\xi,\varrho})_{\mathfrak{m}}$ when $\varrho=\mathbb{1}_v$ is the trivial character.

Remark. — From now on, localization at $\mathfrak{m}$ means that we prescribe the Satake parameters modulo ℓ as usual, but outside $\{v,w\}$.

Let now explain the main steps of the following sections.

(a) Following the arguments of the previous section, we first analyze the torsion cohomology classes of Harris-Taylor perverse sheaves with infinite level at v, and we deduce, cf. Lemma 3.1.17, that, as $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_d(F_v)$, irreducible subquotients of the ℓ-torsion of their cohomology in infinite level at v, with highest non degeneracy level, appear in degrees $0,1$.

(b) In Section 3.2, considering always infinite level at v, we analyze the torsion cohomology classes of the graded parts $\mathrm{gr}^t_!(\Psi_\varrho)$ of the filtration of stratification $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$ and more specifically when $\varrho=\mathbb{1}_v$ is the trivial character. We then deduce, cf. Lemma 3.2.5, that the ℓ-torsion of $H^i(\mathrm{Sh}_{I^v(\infty),\bar s_v},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ does not have, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_v)$, any irreducible generic subquotient whose supercuspidal support is made of characters.

(c) In Section 3.3, we obtain two fundamental results.

– First, cf. Lemma 3.3.6, under the hypothesis that there exist nontrivial torsion cohomology classes, we show that the graded pieces Γ_k of the filtration given by the spectral sequence of vanishing cycles, of the free quotient of $H^0(\mathrm{Sh}_{I^v(\infty),\bar\eta_v},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ are not always given by the lattice Γ_0 of

$$H^0(\mathrm{Sh}_{I,\bar s_v},P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell}((1-t+2\delta)/2))_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell$$

given by the integral cohomology of $P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell}$. Roughly, there exist some k and a short exact sequence $\Gamma_0\hookrightarrow\Gamma_k\twoheadrightarrow T$, where T is nontrivial and torsion.

– We then play with the action of $\mathrm{GL}_d(F_w)$ by allowing infinite level at w. The main observation at the end of the section, cf. Proposition 3.3.13, is that, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, all the irreducible subquotients of the ℓ-torsion of the cokernels T between two lattices in the previous point, up to multiplicities, are also subquotients of the ℓ-torsion of the cohomology of the Shimura variety. In particular, as v and w are playing symmetric roles, these subquotients are not generic, cf. Corollary 3.3.14.

(d) In Section 3.4, the last step is to prove, under the absurd hypothesis that there exist nontrivial torsion cohomology classes while $\bar\rho_{\mathfrak{m}}$ being irreducible, that $S_{\mathfrak{m}}(v)$ contains a full Zelevinsky line modulo ℓ $\{\lambda q_v^n\in\overline{\mathbb{F}}_\ell\mid n\in\mathbb{Z}\}$ which is of order the order of q_v modulo ℓ. As this order is supposed to be strictly greater than d, this is a contradiction. For more insight on the strategy to prove this fact using the previous properties about lattices, we refer to the introduction of Section 3.4.

3.1. Torsion classes for Harris-Taylor perverse sheaves. — We focus on the torsion in the cohomology groups of the Harris-Taylor perverse sheaves $P_\xi(\chi_v, t)_{\overline{\mathbb{Z}}_\ell}$ when the level at v is infinite, and as explained above, cf. the main assumption of Section 2.1, especially when the reduction modulo ℓ of χ_v is the trivial character.

Notation 3.1.1. — We will denote by $I^v \in \mathcal{I}$ a finite level outside v, and we also denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{T}_\xi^{S\cup\{v\}}$ associated to $\mathfrak{m}$, i.e., we do not prescribe the Satake parameters modulo ℓ at v. Let us also set

$$H^i(\mathrm{Sh}_{I^v(\infty),\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}} := \varinjlim_{I_v} H^i(\mathrm{Sh}_{I^vI_v,\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}},$$

which can be viewed as a $\overline{\mathbb{Z}}_\ell[\mathrm{GL}_d(F_v)]$-module. Then, morally, $I^v(\infty)$ is a finite level outside v and infinite at v.

Proposition 3.1.2 (cf. [11], second global result of the introduction)
We have the following resolution of ${}^pj^{=t}_{!*}HT(\chi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell}$

$$\begin{aligned}(3.1.3)\quad 0 \longrightarrow j^{=d}_! HT(\chi_v, \Pi_t\{(t-s)/2\}) \times \mathrm{Speh}_{d-t}(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell} \otimes \Xi^{(s-t)/2} \longrightarrow \cdots \\ \longrightarrow j^{=t+1}_! HT(\chi_v, \Pi_t\{-1/2\} \times \chi_v\{t/2\})_{\overline{\mathbb{Z}}_\ell} \otimes \Xi^{1/2} \\ \longrightarrow j^{=t}_! HT(\chi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell} \longrightarrow {}^pj^{=t}_{!*}HT(\chi_v, \Pi_t)_{\overline{\mathbb{Z}}_\ell} \longrightarrow 0.\end{aligned}$$

Note that

– as this resolution is equivalent to the computation of the sheaves cohomology groups of ${}^pj^{=h}_{!*}HT(\chi_v, \mathrm{St}_h(\chi_v))_{\overline{\mathbb{Z}}_\ell}$ as explained for example in [11, Prop. B.1.5], then, over $\overline{\mathbb{Q}}_\ell$, the proposition follows from the main results of [3].

– Over $\overline{\mathbb{Z}}_\ell$, as every terms are free perverse sheaves, then all maps are necessary strict.

– This resolution, for a a general supercuspidal representation with supercuspidal reduction modulo ℓ, is one of the main result of [11, §2.3]. However, in the case of a character χ_v the arguments are much easier.

Consider the finite level $I^v(n) = I^v I_{v,n}$, where

$$I_{v,n} = \mathrm{Ker}(\mathrm{GL}_d(\mathcal{O}_v) \twoheadrightarrow \mathrm{GL}_d(\mathcal{O}_v/\varpi_v^n)).$$

The strata $\mathrm{Sh}^{\geqslant h}_{I^v(n),\overline{s}_v,\overline{1_h}}$ are smooth, then, cf. the proof of Lemma 2.2.5, the constant sheaf, up to shift, is perverse and so equal to the intermediate extension of the constant sheaf, shifted by $d-h$, on $\mathrm{Sh}^{=h}_{I^v(n),\overline{s}_v,\overline{1_h}}$. In particular, as a constant sheaf, its sheaf cohomology groups are well-known, so, over $\mathrm{Sh}^{\geqslant h}_{I^v(n),\overline{s}_v,\overline{1_h}}$ and so for $\mathrm{Sh}^{\geqslant h}_{I^v(\infty),\overline{s}_v,\overline{1_h}}$, the resolution is completely obvious for ${}^pj^{=h}_{\overline{1_h},!*}HT(\chi_v, \mathrm{St}_h(\chi_v))_{1,\overline{\mathbb{Z}}_\ell}$ if one remembers that $\mathrm{Speh}_i(\chi_v)$ is just the character $\chi_v \circ \det$ of $\mathrm{GL}_i(F_v)$.

The stated resolution is then simply the induced version of the resolution of ${}^pj^{=h}_{\overline{1_h},!*}HT(\chi_v, \mathrm{St}_h(\chi_v))_{1,\overline{\mathbb{Z}}_\ell}$: recall that a direct sum of intermediate extensions is still an intermediate extension.

By the adjunction property, the map

$$(3.1.4)\quad j_!^{=t+\delta}HT(\chi_v,\Pi_t\{-\delta/2\})\times \mathrm{Speh}_\delta(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{\delta/2} \longrightarrow j_!^{=t+\delta-1}HT(\chi_v,\Pi_t\{(1-\delta)/2\})\times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{(\delta-1)/2}$$

is given by

$$(3.1.5)\quad HT(\chi_v,\Pi_t\{-\delta/2\}\times \mathrm{Speh}_\delta(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{\delta/2}\longrightarrow {}^p i^{t+\delta,!}j_!^{=t+\delta-1}HT(\chi_v,\Pi_t\{(1-\delta)/2\})\times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{(\delta-1)/2}.$$

We then have

$$(3.1.6)\quad {}^p i^{t+\delta,!}j_!^{=t+\delta-1}HT(\chi_v,\Pi_t\{(1-\delta)/2\})\times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{(\delta-1)/2} \simeq HT\Big(\chi_v,\Pi_t\{(1-\delta)/2\})\times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\})\times\chi_v\{(\delta-1)/2\}\big)\{t/2\}\Big)_{\overline{\mathbb{Z}}_\ell}\otimes \Xi^{\delta/2}.$$

Indeed one can compute ${}^p i^{h+1,!}j_!^{=h}HT(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell}$ by means of the spectral sequence associated to the exhaustive filtration of stratification of $j_!^{=h}HT(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell}$

$$(3.1.7)\quad (0)=\mathrm{Fil}^0(\chi_v,h)\hookrightarrow \mathrm{Fil}^{-d}(\chi_v,h)\hookrightarrow\cdots \hookrightarrow \mathrm{Fil}^{-h}(\chi_v,h)=j_!^{=h}HT(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell}$$

with graded parts, using Lemma 2.2.5 and [6],

$$\mathrm{gr}^{-k}(\chi_v,h)\simeq {}^p j_{!*}^{=k}HT(\chi_v,\Pi_h\{(h-k)/2\}\otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))_{\overline{\mathbb{Z}}_\ell}((h-k)/2).$$

As remarked before, the sheaf cohomology groups of

$$i^{h+1,*}\,{}^p j_{!*}^{h+k}HT(\chi_v,\Pi_h\{-k/2\}\times \mathrm{St}_k(\chi_v)(h/2))_{\overline{\mathbb{Z}}_\ell}$$

are torsion-free, so, by Grothendieck-Verdier duality, the same is true for

$$i^{h+1,!}\,{}^p j_{!*}^{h+k}HT(\chi_v,\Pi_h\{-k/2\}\times \mathrm{St}_k(\chi_v)(h/2))_{\overline{\mathbb{Z}}_\ell}.$$

The statement follows then from the fact that, over $\overline{\mathbb{Q}}_\ell$, the previous spectral sequence degenerates at E_1.

Remark. — This property is also true when we replace the character χ_v by any irreducible cuspidal representation π_v, cf. [11].

Fact. — In particular, up to homothety, the map (3.1.6), and so those of (3.1.5), is unique. Finally, as the maps of (3.1.3) are strict, the given maps (3.1.4) are uniquely determined, that is, if we forget the infinitesimal parts these maps are independent of the chosen t in (3.1.3).

We now copy the arguments of Section 2.1.

Notation 3.1.8. — For every h such that $1\leqslant h\leqslant d$, let us denote by $i_{I^v}(h,\chi_v)$ the smallest index i such that $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v},{}^p j_{!*}^{=h}HT_\xi(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ has nontrivial torsion: if it doesn't exists, then set $i_{I^v}(h,\chi_v)=+\infty$.

Remark. — By duality, as ${}^p j_{!*}^{=h} = {}^{p+} j_{!*}^{=h}$ for Harris-Taylor local systems associated to a character, note that when $i_I(h,\chi_v)$ is finite then $i_{I^v}(h,\chi_v) \leqslant 0$.

Notation 3.1.9. — Suppose there exists $I \in \mathfrak{I}$ such that there exists h with $1 \leqslant h \leqslant d$ and with $i_{I^v}(h,\chi_v)$ finite, and denote by $h_0(I^v,\chi_v)$ the biggest such h.

Lemma 3.1.10. — *For $1 \leqslant h \leqslant h_0(I^v,\chi_v)$ then we have*

$$i_{I^v}(h,\chi_v) = h - h_0(I^v,\chi_v).$$

Moreover Frob_v *acts by* $\chi_v(\mathrm{Frob}_v) q_v^{(h_0(I^v)+1-h)/2}$.

Proof. — Note first that for every h such that $h_0(I^v,\chi_v) \leqslant h \leqslant s$, the cohomology groups of $j_!^{=h} HT_\xi(\chi_v,\Pi_h)$ are torsion-free. The spectral sequence associated to the filtration (3.1.7), localized at $\mathfrak{m}$, is then concentrated in middle degree and is torsion-free.

Consider then the spectral sequence associated to the resolution (3.1.3): its E_1 terms are torsion-free and it degenerates at E_2. As, by hypothesis, the abutment of this spectral sequence is free and is equal to only one E_2 terms, we deduce that all the maps

$$\begin{aligned} (3.1.11)\quad & H^0\big(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v,\Pi_h\{-\delta/2\}) \times \mathrm{Speh}_\delta(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell} \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \\ & \longrightarrow H^0\big(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v,\Pi_h\{(1-\delta)/2\}) \\ & \qquad\qquad \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\}))_{\overline{\mathbb{Z}}_\ell} \otimes \Xi^{(\delta-1)/2}\big)_{\mathfrak{m}} \end{aligned}$$

are strict. Then from the previous fact stressed after (3.1.6), this property remains true when we consider the associated spectral sequence for $1 \leqslant h' \leqslant h_0(I^v,\chi_v)$.

Consider now $h = h_0(I^v,\chi_v)$, where we know the torsion to be nontrivial. From what was observed above we then deduce that the map

$$\begin{aligned} (3.1.12)\quad & H^0\Big(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, \\ & j_!^{=h_0(I^v,\chi_v)+1} HT_\xi\big(\chi_v,\Pi_{h_0(I^v,\chi_v)}\{-1/2\}\big) \times \chi_v\big\{h_0(I^v,\chi_v)/2\big\}\big)_{\overline{\mathbb{Z}}_\ell} \otimes \Xi^{1/2}\Big)_{\mathfrak{m}} \\ & \longrightarrow H^0\big(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j_!^{=h_0(I^v,\chi_v)} HT_\xi(\chi_v,\Pi_{h_0(I^v,\chi_v)})_{\overline{\mathbb{Z}}_\ell}\big)_{\mathfrak{m}} \end{aligned}$$

has a nontrivial torsion cokernel so that $i_I(h_0(I^v,\chi_v)) = 0$.

Finally for any $1 \leqslant h \leqslant h_0(I^v,\chi_v)$, the map like (3.1.12) for $h+\delta-1 < h_0(I^v,\chi_v)$ are strict so that the $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ are zero for $i < h - h_0$ while when $h+\delta-1 = h_0$ its cokernel has nontrivial torsion, which gives then a nontrivial torsion class in $H^{h-h_0}(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,\Pi_h)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$. □

Lemma 3.1.13. — *The integers $h_0(I^v,\chi_v)$ and $i_{I^v}(h,\chi_v)$ only depend on the reduction modulo ℓ of χ_v.*

Proof. — For P a torsion-free $\overline{\mathbb{Z}}_\ell$-perverse sheaf, recall the well-known short exact sequence

$$0 \longrightarrow H^i(X,P) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \longrightarrow H^i(X, P \otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell) \longrightarrow H^{i+1}(X,P)[\ell] \longrightarrow 0.$$

We apply it to $X=\mathrm{Sh}_{I^v,\overline{s}_v}$ and $P=P_\xi(h_0(I^v,\chi_v),\chi_v)_{\overline{\mathbb{Z}}_\ell}$. Recall that, thanks to our hypothesis on $\mathfrak{m}$, the $\overline{\mathbb{Q}}_\ell$-cohomology of P, localized at $\mathfrak{m}$, is concentrated in degree 0. The same is true for $P_\xi(h_0(I^v,\chi_v),\chi'_v)_{\overline{\mathbb{Z}}_\ell}$ for any character $\chi'_v\equiv\chi_v \bmod \ell$. Recall also that $P_\xi(h_0(I^v,\chi_v),\chi_v)_{\overline{\mathbb{Z}}_\ell}$ (resp. $P_\xi(h_0(I^v,\chi_v),\chi'_v)_{\overline{\mathbb{Z}}_\ell}$) is a local system on the stratum $\mathrm{Sh}^{\geqslant h_0(I^v,\chi_v)}_{I^v,\overline{s}_v}$ so that

$$P_\xi(h_0(I^v,\chi_v),\chi_v)_{\overline{\mathbb{Z}}_\ell}\otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell\simeq P_\xi(h_0(I^v,\chi_v),\chi'_v)_{\overline{\mathbb{Z}}_\ell}\otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell$$

is simply the reduction modulo ℓ of this local system. From the definition of $h_0(I^v,\chi_v)$, the cohomology of $P_\xi(h_0(I^v,\chi_v),\chi_v)_{\overline{\mathbb{Z}}_\ell}$ has torsion in degrees 0 and 1 so that its $\overline{\mathbb{F}}_\ell$-cohomology, localized at $\mathfrak{m}$, is concentrated in degrees $-1,0,1$: moreover, for $h>h_0(I^v,\chi_v)$ the $\overline{\mathbb{F}}_\ell$-cohomology of $P_\xi(h,\chi_v)_{\overline{\mathbb{Z}}_\ell}$ is concentrated in degree 0.

The same is then true for the $\overline{\mathbb{F}}_\ell$-cohomology, localized at $\mathfrak{m}$, of $P_\xi(,h_0(I^v,\chi_v),\chi'_v)_{\overline{\mathbb{Z}}_\ell}$ so that its $\overline{\mathbb{Z}}_\ell$-cohomology, localized at $\mathfrak{m}$, must have torsion in degrees 0 and 1. Moreover, for $h>h_0(I^v,\chi_v)$, the $\overline{\mathbb{F}}_\ell$-cohomology of $P_\xi(h,\chi'_v)_{\overline{\mathbb{Z}}_\ell}$ is concentrated in degree 0. By definition we then have $h_0(I^v,\chi'_v)=h_0(I^v,\chi'_v)$.

Concerning $i_{I^v}(h,\chi'_v)$ the result then follows from the previous lemma. □

From the main assumption of Section 2.1 and as explained in the introduction of this section, we focus on $\Psi_{\xi,\varrho}$ when $\varrho=\mathbb{1}_v$ is the trivial character. We are then interested in the characters χ_v congruent to the trivial character $\mathbb{1}_v$ modulo ℓ.

Notation 3.1.14. — Following the notation of Proposition 2.1.4, we will denote $h_0(I^v)$ (resp. $i_{I^v}(h)$) for $h_0(I^v,\mathbb{1}_v)$ (resp. $i_{I^v}(h,\mathbb{1}_v)$).

Lemma 3.1.15. — *With the notation of Proposition* 2.1.4*, we have* $h_0(I^v)\geqslant h_0(I)$.

Proof. — Consider the previous map (3.1.12) by replacing $h_0(I^v)$ by $h_0(I)$. As by hypothesis the order of q_v modulo ℓ is strictly greater than d, then the pro-order of the local component I_v of I at v is invertible modulo ℓ, so that the functor of invariants under I_v is exact. Note then that, as the I_v-invariants of the map (3.1.12) when replacing $h_0(I^v)$ by $h_0(I)$, has a cokernel which is not free, then the cokernel of (3.1.12), for $h_0(I)$, is also not free. □

From the previous proof, we also deduce that all cohomology classes of any of the $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v},P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ come from the non strictness of some map (3.1.12) with $\Pi_v:=\mathrm{St}_t(\chi_v)$. In the following we will focus on $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v},P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}[\ell]$ as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_v)$. More precisely we are interested in irreducible such subquotients which have maximal non-degeneracy level at v.

Notation 3.1.16. — Fix such a non degeneracy level $\underline{\lambda}$ for $\mathrm{GL}_d(F_v)$ in the sense of Notation 1.1.9, which is maximal for torsion classes in $H^0(\mathrm{Sh}_{I^v(\infty),\overline{s}_v},P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}[\ell]$ for various t such that $1\leqslant t\leqslant d$ and $\chi_v\in\mathrm{Cusp}_{\mathbb{1}_v}(-1)$.

Lemma 3.1.17. — *Let* $\chi_v\in\mathrm{Cusp}_{-1}(\mathbb{1}_v)$*. Then all* $\overline{\mathbb{F}}_\ell[\mathrm{GL}_d(F_v)]$*-irreducible subquotients of* $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v},P_\xi(t,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}[\ell]$*, for* $i\neq 0,1$*, have a level of non degeneracy strictly less than* $\underline{\lambda}$.

Proof. — It easily follows from the observation that the level of non degeneracy of the reduction modulo ℓ of $\mathrm{Speh}_h(\chi_v) \simeq \chi_v$ is strictly less than those of the reduction modulo ℓ of $\mathrm{St}_h(\chi_v)$ which, cf. [5], is irreducible as the order of q_v modulo ℓ is strictly greater than d and so strictly greater than h. □

3.2. Global torsion and genericity. — Recall that $v \in \mathrm{Spl}$ is such that the order of q_v modulo ℓ is strictly greater than d. Let us denote by I^v the component of I outside v. We then simply denote by Ψ_v and $\Psi_{v,\xi}$, the inductive system of perverse sheaves indexed by the finite level $I^v I_v \in \mathfrak{I}$ for varying I_v.

For $\pi_v \in \mathrm{Cusp}_\varrho$, let us denote by $\mathrm{Fil}^1_{!,\pi_v}(\Psi_\varrho)$ the quotient of $\mathrm{Fil}^1_!(\Psi_\varrho)$ such that $\mathrm{Fil}^1_{!,\pi_v}(\Psi_v) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \simeq \mathrm{Fil}^1_!(\Psi_{\pi_v})$ where Ψ_{π_v} is the direct factor of $\Psi_v \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ associated to π_v, cf. [6].

Remark. — In the following, we will be mainly concerned with the case where π_v is a character χ_v whose reduction modulo ℓ is the trivial character. We will then write the main statement in this case.

Recall the following resolution of $\mathrm{Fil}^1_{!,\chi_v}(\Psi_\varrho)$

$$\begin{aligned}(3.2.1)\quad 0 &\longrightarrow j^{=d}_! HT(\chi_v, \mathrm{Speh}_d(\chi_v))_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v((d-1)/2)) \\ &\longrightarrow j^{=d-1}_! HT(\chi_v, \mathrm{Speh}_{d-1}(\chi_v))_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v((d-2)/2)) \longrightarrow \cdots \\ &\longrightarrow j^{=1}_! HT(\chi_v, \chi_v)_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}^1_{!,\chi_v}(\Psi_\varrho) \longrightarrow 0,\end{aligned}$$

which is proved in [6] over $\overline{\mathbb{Q}}_\ell$. I claim it is also true over $\overline{\mathbb{Z}}_\ell$. Indeed, using Lemma 2.2.5, it is equivalent to the fact the sheaf cohomology of $\mathrm{Fil}^1_{!,\chi_v}(\Psi_\varrho)$ is torsion-free, which follows then from [19], the comparison theorem of Faltings-Fargues cf. [15] and the main theorem of [14].

Remark. — In [11], we prove the same resolution for any irreducible cuspidal representation π_v in place of χ_v.

More generally for $\mathrm{gr}^t_!(\Psi_\varrho) \twoheadrightarrow \mathrm{gr}^t_{!,\chi_v}(\Psi_\varrho)$, we have

$$\begin{aligned}(3.2.2)\quad 0 &\longrightarrow j^{=d}_! HT(\chi_v, LT_{\chi_v}(t-1, d-t))_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v((d-2t+1)/2)) \\ &\longrightarrow j^{=d-1}_! HT(\chi_v, LT_{\chi_v}(t-1, d-t-1))_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v((d-2t)/2)) \longrightarrow \cdots \\ &\longrightarrow j^{=t}_! HT(\chi_v, \mathrm{St}_t(\chi_v))_{\overline{\mathbb{Z}}_\ell} \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{gr}^t_{!,\chi_v}(\Psi_\varrho) \longrightarrow 0.\end{aligned}$$

Finally all the torsion cohomology classes of the $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v}, \mathrm{gr}^t_{!,\chi_v}(\Psi_\varrho))_\mathfrak{m}$ come from the non strictness of the maps

$$\begin{aligned}(3.2.3)\quad &H^0(\mathrm{Sh}_{I^v(\infty),\overline{s}_v}, j^{=h+1}_! HT(\chi_v, \Pi_{h+1})_{\overline{\mathbb{Z}}_\ell})_\mathfrak{m} \\ &\qquad\longrightarrow H^0(\mathrm{Sh}_{I^v(\infty),\overline{s}_v}, j^{=h}_! HT(\chi_v, \Pi_h)_{\overline{\mathbb{Z}}_\ell})_\mathfrak{m},\end{aligned}$$

where (Π_h, Π_{h+1}) is of the shape $\big(LT_{\chi_v}(t-1, h-t), LT_{\chi_v}(t-1, h+1-t)\big)$.

We can then copy the proof of Lemma 3.1.10 which gives us the following statement.

Lemma 3.2.4. — *For every h such that $1 \leqslant h \leqslant h_0(I^v)$, the number $i_I(h) = h - h_0(I^v)$ of Notation* 3.1.8*, is also the lowest integer i such that the torsion of $H^i(\mathrm{Sh}_{I^v(\infty),\overline{s}_v}, \mathrm{gr}^h_{!,\chi_v}(\Psi_{\varrho,\xi}))_{\mathfrak{m}}$ is non zero.*

Lemma 3.2.5. — *For every i, the ℓ-torsion of*

$$H^i(\mathrm{Sh}_{I^v(\infty),\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}},$$

as an $\overline{\mathbb{F}}_\ell[\mathrm{GL}_d(F_v)]$-module, does not have an irreducible generic subquotient whose cuspidal support is made of characters.

Remark. — Note that when the order of q_v modulo ℓ is strictly greater than d, then there is no difference between cuspidal or supercuspidal support made of characters.

Proof. — Recall first that, as by hypothesis $\overline{\rho_{\mathfrak{m}}}$ is irreducible, the $\overline{\mathbb{Q}}_\ell$-version of the spectral sequence (2.3.2) degenerates at E_1 so that in particular all the torsion cohomology classes appear in the E_1 terms. As we are only interested in representations with cuspidal support made of characters, we only have to deal with the perverse sheaves $P(t,\chi_v)_{\overline{\mathbb{Z}}_\ell}$ so that the result follows from the previous maps (3.2.3) and the fact that for any $r > 0$, the reduction modulo ℓ of $LT_{\chi_v}(t-1,r)$ does not admit any irreducible generic subquotient. □

3.3. Torsion and modified lattices. — Recall that we argue by contradiction, assuming there exists a finite level I unramified at the place v, such that the torsion of some of the $H^i(\mathrm{Sh}_{I,\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ is non zero. We then increase the level at v to infinity and define the index $h_0(I^v)$ which might be greater than the index $h_0(I)$ defined in level I, cf. Notation 3.1.14, of the first Harris-Taylor perverse sheaf associated to a character χ_v congruent to $\mathbb{1}_v$ modulo ℓ, with nontrivial torsion cohomology class.

We now come back to a finite level at v with two main objectives: first we want to keep the torsion in the cohomology of $P_\xi(\chi_v,h_0)_{\overline{\mathbb{Z}}_\ell}$ and secondly we intend to simplify the spectral sequence of vanishing cycles.

Remark. — The main reason to go to infinite level at v is to be able to use the notion of level of non degeneracy.

To be able to deal with representations, we fix a place $w \neq v$ with $w \in \mathrm{Spl}(I)$ and verifying the same hypothesis as v, i.e., q_w modulo ℓ is of order strictly greater than d.

Notation 3.3.1. — We then denote as before by $I^w(\infty)$ when the level is infinite at w and $I^{w,v}(\infty)$ when the level is infinite at v and w. We also denote h_0 for $h_0(I^w)$, the highest index when torsion appear in the cohomology of a Harris-Taylor perverse sheaf in infinite level at w and maximal level at v.

Lemma 3.3.2. — *With the notations of* 1.1.12*, $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, P_\xi(h_0,\mathbb{1}_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ has nontrivial torsion classes invariant under* $\mathrm{Iw}_v(d-h_0,1,\dots,1)$.

Proof. — Note first that by our absurd hypothesis in Section 2.1,

$$H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, j^{=h_0}_{!*}HT_\xi(\mathbb{1}_v,\mathrm{Speh}_{h_0}(\mathbb{1}_v))_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$$

has nontrivial torsion classes invariant under the action of $\mathrm{GL}_d(\mathcal{O}_v)$ coming from the map (3.1.12) by putting $\mathrm{Speh}_{h_0+1}(\mathbb{1}_v)$ (resp. $\mathrm{Speh}_{h_0}(\mathbb{1}_v)$) in place of

$$\Pi_{h_0(I^v,\chi_v)}\{-1/2\} \times \chi_v\{h_0(I^v,\chi_v)/2\}$$

(resp. $\Pi_{h_0(I^v,\chi_v)}$). Then by putting $\mathrm{St}_{h_0}(\mathbb{1}_v)$ in place of $\Pi_{h_0(I^v,\chi_v)}$ in (3.1.12), we deduce that $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, P_\xi(h_0,\mathbb{1}_v))_{\mathfrak{m}}$ has nontrivial torsion classes invariant under $\mathrm{Iw}_v(d-h_0,1,\dots,1)$, where, with the notations of 1.1.12, $(d-h_0,1,\dots,1)$ is the dual partition of $(h_0+1,1,\dots,1)$. □

Notation 3.3.3. — We will now consider the following level

$$I^w(h_0) := I^w(\infty)\,\mathrm{Iw}_v(d-h_0,1,\dots,1),$$

which is infinite at w and of Iwahori type at v.

Remark. — Dealing with infinite level at w allows to talk about representations of $\mathrm{GL}_d(F_w)$, while Iwahori type subgroup allows to simplify the spectral sequence. Indeed from Lemma 1.1.12, and using the definition of $LT_{\chi_v}(t,s)$ through an induced representation, cf. Definition 1.1.2, we note that for $h \geqslant h_0+2$, then $LT_{\chi_v}(h,d-h-1)$ does not have any nontrivial vector invariant by $\mathrm{Iw}_v(d-h_0,1,\dots,1)$. Moreover for an irreducible representation π_v of $\mathrm{GL}_{d-h}(F_v)$ with $h \geqslant h_0+1$, then $LT_{\chi_v}(h_0,h-h_0)\times\pi_v$ admits nontrivial invariant vectors by $\mathrm{Iw}_v(d-h_0,1,\dots,1)$ if and only if $\pi_v \simeq \chi_{v,1}\times\cdots\times\chi_{v,d-h}$ is unramified, cf. the examples following Lemma 1.1.12. Similar simplifications also appear in Lemma 3.3.9.

We then focus on the free quotient of $H^0(\mathrm{Sh}_{I^w(h_0),\overline{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell}[d-1])_{\mathfrak{m}}$ by means of the spectral sequence of vanishing cycles. From (2.3.1), we are then lead to study the cohomology of $\Psi_{\xi,\mathbb{1}_v}$ by means of its filtration of stratification and so we first focus on the cohomology of $\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ for a character $\chi_v\in\mathrm{Cusp}_{-1}(\mathbb{1}_v)$ which is by definition a quotient of $\mathrm{gr}^{h_0}_!(\Psi_{\xi,\mathbb{1}_v})$. To do so, consider first the filtration constructed in [6]:

$$\mathrm{Fil}^{d-h_0}(\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})) \subset \cdots \subset \mathrm{Fil}^0(\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})),$$

with successive free graded parts $\mathrm{gr}^i(\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}))$ which, for $i\geqslant 0$, is a $\overline{\mathbb{Z}}_\ell$-structure of the $\overline{\mathbb{Q}}_\ell$-perverse sheaf $P(h_0+i,\chi_v)_{\overline{\mathbb{Z}}_\ell}((1-h_0+i)/2)$. We then introduce two $\overline{\mathbb{Z}}_\ell$-lattices of

$$H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, P_\xi(h_0+i,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell. \tag{3.3.4}$$

– The first one denoted by $\Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+i)$ is given by the free $\overline{\mathbb{Z}}_\ell$-cohomology: recall that as the order of q_v modulo ℓ is supposed to be strictly greater than d then, cf. [5], the reduction modulo ℓ of $\mathrm{St}_{h_0+i}(\chi_v)$ and that of $\chi_v[t]_D$, remains irreducible so that, up to homothety, there is a unique stable lattice of $P_\xi(h_0+i,\chi_v)_{\overline{\mathbb{Z}}_\ell}$.

– The spectral sequence associated to the previous filtration of $\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ provides a filtration of $H^0_{\mathrm{free}}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$, and $\Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+i,h_0)$ is then the lattice of the subquotient in this filtration corresponding to (3.3.4).

By construction we have

$$\Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1) \hookrightarrow \Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,h_0), \tag{3.3.5}$$

but the cokernel of torsion might be nontrivial due to torsion in the remaining of the E_∞ terms. The main point is first to show, under the absurd hypothesis of Section 2.1, that this cokernel is nontrivial and then, to prove some property verified by its ℓ-torsion and finally achieve to a contradiction. We then first focus on the first step about nontriviality.

Lemma 3.3.6. — *The cokernel T of* (3.3.5)*:*

$$0 \longrightarrow \Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1) \longrightarrow \Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,h_0) \longrightarrow T \longrightarrow 0,$$

is non zero and every irreducible subquotient of its ℓ-torsion as a $(\mathbb{T}^S_{\xi,\mathfrak{m}} \times \mathrm{GL}_d(F_w)) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$-module, can be obtained as a subquotient of the torsion submodule of the cokernel of

$$H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v, \mathrm{St}_{h_0+1}(\chi_v))_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}} \longrightarrow H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v))_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}. \tag{3.3.7}$$

Proof. — The idea is to compute the cohomology of $\mathrm{gr}_!^{h_0}(\Psi_{\xi,\chi_v})$ in two different ways, first by means of the spectral sequence associated to (3.2.2) and secondly by means of its filtration of stratification with graded parts the Harris-Taylor perverse sheaves.

To argue we will rest on the level of non degeneracy at v so that we pass to $I^{w,v}(\infty)$-level: as q_v modulo ℓ is of order $> d$ taking invariant under any sub-group of $\mathrm{GL}_d(\mathcal{O}_v)$ is an exact functor and it will be easy to go down to level $I^w(h_0)$.

First note that the $I^{w,v}(\infty)$-version of (3.3.7) is non strict if and only if the same is true for its non induced version in the next formula, whatever a representation Π_{h_0} of $\mathrm{GL}_{h_0}(F_v)$ is:

$$H^0(\mathrm{Sh}_{I^{w,v}(\infty),\bar{s}_v,\overline{1_{h_0}}}, j^{=h_0+1}_{\overline{1_{h_0}},!} HT_\xi(\chi_v, \Pi_{h_0} \otimes \chi_v)_{\overline{1_{h_0}},\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}} \longrightarrow H^0(\mathrm{Sh}_{I^{w,v}(\infty),\bar{s}_v,\overline{1_{h_0}}}, j^{=h_0}_{\overline{1_{h_0}},!} HT_\xi(\chi_v, \Pi_{h_0})_{\overline{1_{h_0}},\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}, \tag{3.3.8}$$

where we set

$$j^{=h_0+1}_{\overline{1_{h_0}}} : \mathrm{Sh}^{=h_0+1}_{I^v(\infty),\bar{s}_v,\overline{1_{h_0}}} \hookrightarrow \mathrm{Sh}^{\geqslant 1}_{I^v(\infty),\bar{s}_v},$$

where $\mathrm{Sh}^{=h_0+1}_{I^{w,v}(\infty),\bar{s}_v,\overline{1_{h_0}}}$ is the disjoint union of the pure strata, cf. Notation 2.2.1, $\mathrm{Sh}^{=h_0+1}_{I^{w,v}(\infty),\bar{s}_v,g}$ contained in $\mathrm{Sh}^{\geqslant h_0}_{I^{w,v}(\infty),\bar{s}_v,\overline{1_{h_0}}}$.

Lemma 3.3.9

(i) *With $\Pi_{h_0} = \mathrm{St}_{h_0}(\chi_v)$, the cokernel of the induced version of* (3.3.8) *has nontrivial vectors invariant under* $\mathrm{Iw}_v(d-h_0,1,\dots,1)$.

(ii) *For $h > h_0$, whatever are the representations Π_h and Π_{h+1} of respectively $\mathrm{GL}_h(F_v)$ and $\mathrm{GL}_{h+1}(F_v)$, the cokernel of*

$$H^0(\mathrm{Sh}_{I^{w,v}(\infty),\bar{s}_v}, j_!^{=h+1} HT_\xi(\chi_v, \Pi_{h+1})_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}} \longrightarrow H^0(\mathrm{Sh}_{I^{w,v}(\infty),\bar{s}_v}, j_!^{=h} HT_\xi(\chi_v, \Pi_h)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}, \tag{3.3.10}$$

does not have any non zero invariant vector under $\mathrm{GL}_d(\mathcal{O}_v)$.

(iii) *For* $\Pi_h = LT_{\chi_v}(h_0-1, h-h_0)$ *and* $\Pi_{h+1} = LT_{\chi_v}(h_0-1, h-h_0+1)$*, as in* (3.2.3)*, the cokernel of* (3.3.10) *does not have any non zero invariant vector under* $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$.

Proof. — The integer h_0 is chosen so that, by imposing Π_{h_0} to be unramified, using also the fact that q_v modulo ℓ is of order $> d$ so that the functor of $P_{h_0,d}(\mathcal{O}_v)$-invariants is exact, then the cokernel of (3.3.8) has nontrivial vectors invariant under $P_{h_0,d}(\mathcal{O}_v)$. Then when $\Pi_{h_0} = \mathrm{St}_{h_0}(\chi_v)$, by modifying the factor $\mathrm{GL}_{d-h_0}(\mathcal{O}_v)$ by its classical Iwahori subgroup, we then deduce (i).

(ii) It follows from the definition of h_0 and the fact that the functor of $\mathrm{GL}_d(\mathcal{O}_v)$-invariants is exact.

(iii) If there were nontrivial zero invariant vectors under $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$ then replacing Π_h and Π_{h+1} respectively by $\mathrm{Speh}_h(\chi_v)$ and $\mathrm{Speh}_{h+1}(\chi_v)$, there would exist non zero invariants under $\mathrm{GL}_d(\mathcal{O}_v)$, which is not the case by (ii), cf. the examples following Lemma 1.1.12. □

We then compute the $\mathfrak{m}$-localized cohomology of $\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ in level $I^{w,v}(\infty)$ having nontrivial invariant under $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$. By maximality of h_0, note that for $h_0 < t \leqslant d$, the cohomology groups of

$$P_\xi(t, \chi_v)_{\overline{\mathbb{Z}}_\ell} \quad \text{and} \quad j_!^{=t} HT_\xi(\pi_v, LT_{\chi_v}(h_0-1, t-h_0))_{\overline{\mathbb{Z}}_\ell},$$

after localization by $\mathfrak{m}$, do no have any nontrivial torsion vector invariant under $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$ as explained in the previous lemma.

(1) Following the proof of 3.1.10 with the spectral sequence associated to (3.2.2) and neglecting torsion classes which do not have nontrivial vectors invariant by $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$, we then deduce that $H^i_{\mathrm{tor}}(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, \mathrm{gr}^{h_0}_!(\Psi_{\xi,\chi_v}))_\mathfrak{m}$ does not have any nontrivial vector invariant under $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$ if $i \neq 0, 1$, while for $i = 0$ the torsion is nontrivial and the vectors invariant by $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$ are given by the non strictness of

$$(3.3.11) \quad H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v, LT_{\chi_v}(h_0-1, 1))_{\overline{\mathbb{Z}}_\ell})_\mathfrak{m} \\ \longrightarrow H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v))_{\overline{\mathbb{Z}}_\ell})_\mathfrak{m}.$$

(2) Concerning $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, P_\xi(h_0, \chi_v)_{\overline{\mathbb{Z}}_\ell})_\mathfrak{m}$, its torsion submodule is parabolically induced, so that beside those coming from the non strictness of (3.3.11), there is also the contribution given by the non strictness of (3.3.7), which contains in particular a subquotient, denoted $\widetilde{T}$, such that $\widetilde{T}[\ell]$ is of level of non degeneracy strictly greater than those appearing in (3.3.11). Note moreover that

– $\widetilde{T}[\ell]$ has nontrivial vectors under $\mathrm{Iw}_v(d-h_0, 1, \dots, 1)$;

– as $(\mathbb{T}^S_{\xi,\mathfrak{m}} \times \mathrm{GL}_d(F_w)) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$-modules, the irreducible subquotients of $\widetilde{T}[\ell]$ are also subquotients of the torsion of the ℓ-torsion of the cokernel of (3.3.11). Indeed, $\widetilde{T}[\ell]$ is given as the cokernel of (3.3.11) where we replace $LT_{\chi_v}(h_0-1, 1)$ by $\mathrm{St}_{h_0+1}(\chi_v)$.

(3) Consider then the cohomology of $\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\varrho,\xi})$ computed through its filtration of stratification with graded parts, up to Galois shifts,

$$H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, P_\xi(h_0+k,\chi_v)_{\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}},$$

for $0 \leqslant k \leqslant d-h_0$, and more particularly *the induced filtration of the free quotient of* $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, \mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\mathbb{1}_v,\xi}))_{\mathfrak{m}}$ as before. As the level of non degeneracy of $\widetilde{T}[\ell]$ is higher than that of the ℓ-torsion of $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, \mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_{\mathbb{1}_v,\xi}))_{\mathfrak{m}}$, computed by means of the spectral sequence associated to (3.2.2), they must be graded parts of this filtration. We then have a filtration of the free quotient of $H^0(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, \mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\mathbb{1}_v,\xi}))_{\mathfrak{m}}$ for which, among the graded parts, appear

– torsion modules such as $\widetilde{T}$,

– and the free graded parts which are lattices $\Gamma_{\xi,\chi_v,\mathfrak{m}}(I^v, h_0+i)$ of the free quotient of the localized cohomology of $P_\xi(\chi_v, h_0+i)$ for $0 \leqslant i \leqslant d-h_0$.

We now go back to the level $I^w(h_0) = I^w(\infty)\,\mathrm{Iw}_v(d-h_0,1,\ldots,1)$: as q_v modulo ℓ is of order strictly greater than d, the functor of $\mathrm{Iw}_v(d-h_0,1,\ldots,1)$-invariants is exact. As only the cohomology of $P_\xi(\chi_v, h_0+i)$ for $i=0,1$ contributes, the result follows from the fact that $\widetilde{T}$ has a nontrivial invariant vector under $\mathrm{Iw}_v(d-h_0,1,\ldots,1)$. $\square$

Recall that

$$\mathrm{gr}^{h_0}_!(\Psi_{\xi,\mathbb{1}_v}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\chi_v \in \mathrm{Cusp}_{\mathbb{1}_v}} \mathrm{gr}^{h_0}_!(\Psi_{\xi,\chi_v})$$

so that we can find a filtration of $\mathrm{gr}^{h_0}_!(\Psi_{\xi,\mathbb{1}_v})$ whose graded parts are free and isomorphic, after tensoring with $\overline{\mathbb{Q}}_\ell$, to $\mathrm{gr}^{h_0}_!(\Psi_{\xi,\chi_v})$. Arguing as in the proof of Lemma 3.1.10, using (3.2.3), we have the following result.

Lemma 3.3.12. — *For every $t \geqslant 1$, let $j(t)$ be the minimal integer j such that the torsion of $H^j(\mathrm{Sh}_{I^{w,v}(\infty),\overline{s}_v}, \mathrm{gr}^t_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$ has nontrivial invariant vectors under $\mathrm{Iw}_v(d-h_0,1,\ldots,1)$. Then*

$$j(t) = \begin{cases} +\infty & \text{if } t \geqslant h_0+1, \\ t-h_0 & \text{for } 1 \leqslant t \leqslant h_0. \end{cases}$$

Moreover as a $(\mathbb{T}_{\xi,\mathfrak{m}} \times \mathrm{GL}_d(F_w)) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$-module, up to multiplicities, the irreducible subquotients of $H^{j(t)}_{\mathrm{tor}}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \mathrm{gr}^t_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$ are independent of t.

Proposition 3.3.13. — *Up to multiplicities, the set of irreducible $\overline{\mathbb{F}}_\ell[\mathrm{GL}_d(F_w)]$-subquotients of the ℓ-torsion of*[4] $H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \mathrm{gr}^{h_0}_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$*, are the same as those of $H^{d-h_0}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$.*

Proof. — We compute $H^{d-h_0(I^v)}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ using the filtration $\mathrm{Fil}^\bullet_!(\Psi_{\xi,\mathbb{1}_v})$ through the spectral sequence (2.3.2). Recall that for every $p+q \neq 0$, the free quotient of $E^{p,q}_{!,\varrho,1}$ are zero. By definition of the filtration, these $E^{p,q}_{!,\varrho,1}$ are trivial for $p \geqslant 0$ while, thanks to the previous lemma, for any $p \leqslant -1$ they are zero for $p+q < j(p) := p-h_0$.

[4] or those of $H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, P_\xi(h_0,\chi_v))_{\mathfrak{m}}$ as explained above

Note then that $E^{-1,j(1)+1}_{!,\varrho,1}$, which is torsion and non zero, according to the previous lemma, is equal to $E^{j(1)}_{!,\varrho,\infty}\simeq H^{d-h_0}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$. □

Combining the result of Lemma 3.3.6 with the previous proposition, we then deduce that the cokernel T of Lemma 3.3.6 verifies the following property. As an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, every irreducible subquotient of $T[\ell]$ is also a subquotient of $H^{d-h_0}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$. Then applying Lemma 3.2.5 at the place w, which satisfies the same hypothesis as v, we then deduce the following result.

Corollary 3.3.14. — *As an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, the ℓ-torsion of the cokernel T of Lemma* 3.3.6 *does not contain any irreducible generic subquotient with cuspidal support made of characters.*

We can now repeat the arguments with $\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ for any $1\leqslant k\leqslant h_0$. More precisely, cf. the last remark of Section 2.3, consider $\mathrm{Fil}^i(\mathrm{gr}^k_{!,\chi_c}(\Psi_{\xi,\mathbb{1}_v}))$ for $i=h_0-k$ and $i=h_0-k+2$. As by hypothesis, the torsion of $H^{d-1}(\mathrm{Sh}_{I,\overline{\eta}},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ is nontrivial, then there exists $\widetilde{\mathfrak{m}}\subset\mathfrak{m}$ such that

$$\Pi_{\widetilde{\mathfrak{m}}}\simeq\mathrm{St}_{h_0+1}(\chi_v)\times\chi_{v,1}\times\cdots\times\chi_{v,d-h_0-1},$$

with $\chi_v\equiv\mathbb{1}_v \bmod \ell$. Moreover as before

– $\mathrm{Fil}^{h_0+2-k}(\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}))$ has trivial cohomology groups in level $I^w(h_0)$ because the irreducible constituents of $\mathrm{Fil}^{h_0+2-k}(\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}))\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell$ are, up to Galois shift, Harris-Taylor perverse sheaves $P(t,\chi_v)$ with $t\geqslant h_0+2$;

– we can apply the previous argument relatively to $\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ to the quotient

$$Q:=\mathrm{Fil}^{h_0-k}(\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}))/\,\mathrm{Fil}^{h_0+2-k}(\mathrm{gr}^{h_0}_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})),$$

so that, denoting by $\Gamma'_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k)$ the lattice of (3.3.4) given by the free quotient of $H^0(\mathrm{Sh}_{I^w(\infty),\overline{s}_v},Q)_{\mathfrak{m}}$, the cokernel T'_k of

$$0\longrightarrow\Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1)\longrightarrow\Gamma'_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k)$$

is such that $T'_k[\ell]\neq(0)$ and, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, it does not contain any irreducible generic subquotient made of characters.

In addition of the previous arguments, we also have to deal with the torsion in the cohomology groups of

$$\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})/\,\mathrm{Fil}^{h_0-k}(\mathrm{gr}^k_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})),$$

which could modify the lattice $\Gamma'_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k)$ to give the good one denoted above by $\Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k)$. Note again that, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, this ℓ-torsion does not contain any irreducible generic subquotient made of characters, so the cokernel of

$$\Gamma'_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k)\hookrightarrow\Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k),$$

is again such that, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, its ℓ-torsion does not contain any irreducible generic subquotient made of characters.

Forgetting again Galois shifts, we then conclude that the ℓ-torsion of the cokernel T_k of

$$0 \longrightarrow \Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1) \longrightarrow \Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,k) \tag{3.3.15}$$

is non zero and, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, it does not contain any irreducible generic subquotient made of characters.

We now compute $H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},\Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$ by means of the spectral sequence of vanishing cycles using the filtration

$$\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \cdots \lhook\joinrel\twoheadrightarrow \mathrm{Fil}^{d-1}_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \mathrm{Fil}^{d}_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \Psi_{\xi,\mathbb{1}_v}.$$

Recall that we can filter each of the $\mathrm{gr}^k_!(\Psi_{\xi,\mathbb{1}_v})$ such that the graded parts are, after tensoring with $\overline{\mathbb{Q}}_\ell$ and up to Galois shift, of the form $P_\xi(\chi_v,t)$ with $\chi_v\in\mathrm{Cusp}_{-1}(\mathbb{1}_v)$. As before, arguing by contradiction, we suppose that the torsion of $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_\ell})_{\mathfrak{m}}$ is nontrivial, and we pay special attention to the lattices of

$$V_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1)(\delta) := H^i(\mathrm{Sh}_{I,\bar{s}_v},P_\xi(h_0+1,\chi_v))_{\overline{\mathbb{Z}}_\ell}(\delta)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell, \tag{3.3.16}$$

for $\chi_v\in\mathrm{Cusp}_{-1}(\mathbb{1}_v)$ and various δ.

(a) We first start with $\delta=-h_0/2$. Note that by our choice of Iwahori subgroup, $H^i(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},\Psi_{\xi,\mathbb{1}_v}/\,\mathrm{Fil}^{h_0+1}_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$ are all zero. Indeed the torsion-free graded parts $\mathrm{grr}^k(\Psi_{\xi,\mathbb{1}_v})$ of any exhaustive filtration of $\Psi_{\xi,\mathbb{1}_v}/\,\mathrm{Fil}^{h_0+1}_!(\Psi_{\xi,\mathbb{1}_v})$, up to Galois torsion, are such that $\mathrm{grr}^k(\Psi_{\xi,\mathbb{1}_v})\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell\simeq P(t,\chi_v)$ with $\chi_v\in\mathrm{Cusp}_{-1}(\mathbb{1}_v)$ and $t\geqslant h_0+2$. Then every irreducible constituent of $H^i(\mathrm{Sh}_{I^{w,v}(\infty),\bar{s}_v},\mathrm{grr}^k(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell$, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_v)$ is a subquotient of an induced representation $r_\ell(\mathrm{St}_t(\chi_v))\{\delta/2\}\times\tau$ for some irreducible $\overline{\mathbb{F}}_\ell$-representation τ of $\mathrm{GL}_{d-t}(F_v)$. In particular such a representation does not have nontrivial invariants under $\mathrm{Iw}_v(d-h_0,1,\dots,1)$, so that, as the functor of $\mathrm{Iw}_v(d-h_0,1,\dots,1)$-invariants is exact, there is no cohomology in level $I^w(h_0)$ as stated.

We then deduce that $H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},P_{\xi,\overline{\mathbb{Q}}_\ell}(h_0+1,\chi_v))(\delta)_{\mathfrak{m}}$ is a quotient of $H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},\Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell$ and we denote by

$$\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,+)$$

its stable lattice induced by $H^0_{\mathrm{free}}(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},\Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$.

Proposition 3.3.17. — *With the previous notations, we have an exact sequence*

$$0\longrightarrow\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,+)\longrightarrow\Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1)\longrightarrow T\longrightarrow 0,$$

where $T[\ell]$, as a $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, does not have any irreducible generic subquotient with cuspidal support made of characters.

Proof. — We compute $H^0(\mathrm{Sh}_{I^w(h_0),\bar{s}_v},\Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$ by means of the spectral sequence associated to the filtration

$$\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \cdots \lhook\joinrel\twoheadrightarrow \mathrm{Fil}^{d-1}_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \mathrm{Fil}^{d}_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \Psi_{\xi,\mathbb{1}_v}.$$

Recall that $H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}} = H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \mathrm{Fil}_!^{h_0+1}(\Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$. Let us denote

$$K_0 = \mathrm{Ker}(\mathrm{Fil}_!^{h_0+1}(\Psi_{\xi,\mathbb{1}_v}) \twoheadrightarrow P_\xi(h_0+1,\chi_v)(-h_0/2)).$$

Recall that, over $\overline{\mathbb{Q}}_\ell$, all the cohomology groups are concentrated in degree zero. We then have an exact sequence

$$0 \longrightarrow \Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1,+) \longrightarrow \Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1) \longrightarrow T \longrightarrow 0,$$

where $T \hookrightarrow H^1_{\mathrm{tor}}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, K_0)_{\mathfrak{m}}$. The statement about $T[\ell]$ then follows from the previous section. □

(b) Consider now the case $\delta = h_0/2$ and denote as before by the lattice

$$\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-)$$

of $V_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1)(h_0/2)$ induced by the free quotient of $H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$. Consider $\mathrm{Fil}^1_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}) \lhook\joinrel\twoheadrightarrow \Psi_{\xi,\mathbb{1}_v}$.

Remark. — Before $\mathrm{Fil}^1_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$ was defined as a quotient of $\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v})$. To separate the characters $\chi_v \in \mathrm{Cusp}_{-1}(\mathbb{1}_v)$ in $\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v})$ we start with a filtration of $j^{=1,*}\,\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v})$ with

$$j^{=1,*}\,\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\chi_v \in \mathrm{Cusp}_{-1}(\mathbb{1}_v)} HT_{\xi,\overline{\mathbb{Q}}_\ell}(\chi_v,\chi_v),$$

by considering a numbering of $\mathrm{Cusp}_{-1}(\mathbb{1}_v)$. In particular for a fixed χ_v, when it appears in first or last position, we may have to modify the lattice of $HT(\chi_v,\chi_v)$. But as, up to homothety, $HT(\chi_v,\chi_v)$ has a unique stable lattice, we then obtain the same $\mathrm{Fil}^1_{!,\chi_v}(\Psi_{\xi,\mathbb{1}_v})$.

Proposition 3.3.18. — *The lattices $\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-)$ and $\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,+)$ are not isomorphic.*

Proof. — Recall that the $\mathfrak{m}$-cohomology of the perverse sheaves $P_\xi(t,\chi_v)((t-1)/2)$ for $t \geqslant h_0+2$, does not have nontrivial invariant vectors under $\mathrm{Iw}_v(d-h_0,1,\dots,1)$, then

$$H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, P_\xi(h_0+1,\chi_v)_{\overline{\mathbb{Z}}_\ell}(h_0/2))_{\mathfrak{m}} \lhook\joinrel\longrightarrow H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}},$$

and so the lattice $\Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,1)$ is such that we have an exact sequence

$$0 \longrightarrow \Gamma_{\xi,\chi_v,\mathfrak{m}}(I,h_0+1) \longrightarrow \Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,1) \longrightarrow T \longrightarrow 0, \tag{3.3.19}$$

where $T \neq (0)$ is such that $T[\ell]$, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, does not contain any irreducible generic subquotient made of characters. By construction we also have

$$\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-) = \Gamma_{\xi,\chi_v,!,\mathfrak{m}}(I,h_0+1,1),$$

so by composing the map of Proposition 3.3.17 and (3.3.19), we obtain an exact sequence

$$0 \longrightarrow \Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,+) \longrightarrow \Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-) \longrightarrow T_1 \longrightarrow 0,$$

where the ℓ-torsion T_1, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, does not contain any irreducible generic subquotient made of characters. We then conclude from the fact that the modulo ℓ reduction of these two lattices admits, as an $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(F_w)$, a generic subquotient. □

3.4. Global lattices and generic representations. — We argue on the set $S_{\mathfrak{m}}(v)$ of eigenvalues modulo ℓ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$. By hypothesis there exists $\lambda := \chi_v(\varpi_v)$ such that

$$\{\lambda q_v^{h_0/2}, \lambda q_v^{h_0/2-1}, \dots, \lambda q_v^{-h_0/2}\} \subset S_{\mathfrak{m}}(v),$$

where $h_0 \geqslant 1$ is defined above, under the hypothesis that there exists nontrivial torsion. More precisely, $\lambda q_v^{h_0/2}$ is the reduction modulo ℓ of the eigenvalue of Frob_v acting on $P_\xi(\chi_v, h_0+1)_{\overline{\mathbb{Z}}_\ell}(h_0/2)$ such that the torsion of $H^0(\mathrm{Sh}_{I,\overline{s}_v}, P_\xi(\chi_v, h_0)_{\overline{\mathbb{Z}}_\ell}(h_0/2))_{\mathfrak{m}}$ is non zero.

Let us explain the strategy which will be developed in the following. We first start with $\lambda_0 q_v^{h_0/2} \in S_{\mathfrak{m}}(v)$ and we want to prove that $S_{\mathfrak{m}}(v)$ contains a subset $\{\lambda_1 q_v^{h_0/2}, \lambda_1 q_v^{h_0/2-1}, \dots, \lambda_1 q_v^{-h_0/2}\}$ such that

– there exists r with $0 < r < h_0 + 1$ with $\lambda_1 q_v^{h_0/2} = \lambda_0 q_v^{h_0/2+r}$,

– and $\lambda_1 q_v^{h_0/2}$ is the reduction modulo ℓ of the eigenvalue of Frob_v acting on $P_\xi(\chi'_v, h_0+1)_{\overline{\mathbb{Z}}_\ell}(h_0/2)$ such that the torsion of $H^0(\mathrm{Sh}_{I,\overline{s}_v}, P_\xi(\chi'_v, h_0)_{\overline{\mathbb{Z}}_\ell}(h_0/2))_{\mathfrak{m}}$ is non zero.

We then obtain another interval inside $S_{\mathfrak{m}}(v)$ strictly containing the previous one and we can then play again with $\lambda_1 q_v^{h_0}$ and repeat the above property. At the end we then obtain the full set $\{\lambda_0 q_v^n \mid n \in \mathbb{Z}\}$ which is of order the order of q_v modulo ℓ. But this order is by hypothesis strictly greater than d although trivially the set $S_{\mathfrak{m}}(v)$ of eigenvalues of $\rho(\mathrm{Frob}_v)$ is of order $\leqslant d$.

We now explain how to increase the interval as stated above. To do so, start first with a classical fact concerning $\overline{\mathbb{Z}}_\ell[G]$-modules, where G is a group. Let then Γ, Γ_1 and Γ_2 be three $\overline{\mathbb{Z}}_\ell$-free modules with an action of a group G such that we have an exact sequence

$$0 \longrightarrow \Gamma_1 \longrightarrow \Gamma \longrightarrow \Gamma_2 \longrightarrow 0, \tag{3.4.1}$$

which is G equivariant. We then suppose that this extension is split over $\overline{\mathbb{Q}}_\ell$, i.e.,

$$\Gamma \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \simeq (\Gamma_1 \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell) \oplus (\Gamma_2 \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell).$$

Let then set $\Gamma'_2 := \Gamma \cap (\Gamma_2 \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell)$ and $\Gamma'_1 := \Gamma/\Gamma'_2$ so that the sequence

$$0 \longrightarrow \Gamma'_2 \longrightarrow \Gamma \longrightarrow \Gamma'_1 \longrightarrow 0$$

is exact. We then have the following commutative diagram

$$
\begin{array}{ccccc}
 & & \Gamma_1 & = & \Gamma_1 \\
 & & \downarrow & & \downarrow \\
\Gamma_2' & \hookrightarrow & \Gamma & \twoheadrightarrow & \Gamma_1' \\
\| & & \downarrow & & \downarrow \\
\Gamma_2' & \hookrightarrow & \Gamma_2 & \twoheadrightarrow & T,
\end{array}
\tag{3.4.2}
$$

where T is of torsion and zero if and only if (3.4.1) is split, i.e., $\Gamma_1' = \Gamma_1$ and $\Gamma_2' = \Gamma_2$.

Important remark. — In the following we will consider $\mathbb{T}_{I,\xi,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-free modules. Note that when $\Gamma_i \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ for $i = 1, 2$ are isotypic for the Galois action relatively to a character $\overline{\chi}_i$ such that $\overline{\chi}_1 \not\simeq \overline{\chi}_2$, then from the previous diagram, we have $T = 0$ and $\Gamma \simeq \Gamma_1 \oplus \Gamma_2$.

The idea is to consider two distinct filtrations $\Gamma := H^0_{\mathrm{free}}(\mathrm{Sh}_{I^w(h_0),\overline{s}_v}, \Psi_{\xi,\mathbb{1}_v})_{\mathfrak{m}}$ and we have to be able to go from one filtration to the other one by using repeatedly the previous diagram. The aim of this section is to explain that, upon the hypothesis that the torsion is non zero, such a process is impossible.

(a) First filtration: recall that

$$\Gamma \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\widetilde{\mathfrak{m}}} \Pi^I_{\widetilde{\mathfrak{m}}} \otimes \rho_{\widetilde{\mathfrak{m}}},$$

so that, by fixing any numbering

$$\{\widetilde{\mathfrak{m}} \subset \mathfrak{m} \mid \Pi^I_{\widetilde{\mathfrak{m}}} \neq (0)\} = \{\widetilde{\mathfrak{m}}_1, \dots, \widetilde{\mathfrak{m}}_r\},$$

we define a filtration $\mathrm{Fil}^\bullet(\Gamma)$ with graded parts $\mathrm{gr}^k(\Gamma)$ which is a stable lattice of $\Pi^I_{\widetilde{\mathfrak{m}}_k} \otimes \rho_{\widetilde{\mathfrak{m}}_k}$. With the previous notations, we suppose that

$$\Pi_{\widetilde{\mathfrak{m}}_r,v} \simeq \mathrm{St}_{h_0+1}(\chi_v) \times \chi_{v,1} \times \cdots \times \chi_{v,d-h_0-1}$$

with $\chi_v \in \mathrm{Cusp}_{-1}(\mathbb{1}_v)$. As the reduction modulo ℓ of $\rho_{\widetilde{\mathfrak{m}}_r}$ is irreducible, then $\mathrm{gr}^r(\Gamma)$ is typical, in the sense of [21, §5], i.e.,

$$\mathrm{gr}^r(\Gamma) \simeq \Gamma_r \otimes \Lambda_r,$$

where Γ_r is a stable lattice of $\Pi_{\widetilde{\mathfrak{m}}_r}$ on which the Galois action is trivial and Λ_r is a stable lattice of $\rho_{\widetilde{\mathfrak{m}}_r}$.

(b) The second filtration of Γ is the one induced by the filtration $\mathrm{Fil}^\bullet_!(\Psi_{\xi,\mathbb{1}_v})$ and we try, using diagrams like those above, to end up to the previous lattice $\mathrm{gr}^r(\Gamma) \simeq \Gamma_r \otimes \Lambda_r$. As explained in the previous section, $\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I, h_0+1, +)$ is a quotient of Γ and so a quotient of $\mathrm{gr}^r(\Gamma)$ so that

$$\Gamma_r \simeq \Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I, h_0+1, +).$$

At the opposite, we see, cf. the remark before Proposition 3.3.18, $\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0{+}1,-)$ as a quotient of $H^0_{\mathrm{free}}(\mathrm{Sh}_{I,\overline{s}_v},\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$. From Proposition 3.3.18,

$$\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-) \quad \text{and} \quad \Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,+)$$

are not isomorphic.

We can now apply the process of Diagram (3.4.2) with the successive irreducible subquotients of $H^0(\mathrm{Sh}_{I^w(h_0),\overline{s}_v},\Psi_{\xi,\mathbb{1}_v}/\mathrm{Fil}^1_!(\Psi_{\xi,\mathbb{1}_v}))_{\mathfrak{m}}$ until $\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-)$ is modified during the exchange from subspace to quotient. From Proposition 3.3.18, the typical properties coming from the irreducibility of $\rho_{\mathfrak{m}}$, we then deduce that there should exist $\chi'_v\in\mathrm{Cusp}_{-1}(\varrho)$ and a diagram (3.4.2) with $\Gamma'_2=\Gamma_{\xi,\chi_v,\Psi,\mathfrak{m}}(I,h_0+1,-)$ and Γ'_1 associated to a subquotient of

$$H^0\Big(\mathrm{Sh}_{I,\overline{s}_v},P_\xi(t,\chi'_v)_{\overline{\mathbb{Z}}_\ell}\Big(\frac{t-1}{2}-\delta\Big)\Big)_{\mathfrak{m}}.$$

Note that, as this $P_\xi(t,\chi'_v)_{\overline{\mathbb{Z}}_\ell}((t-1)/2-\delta)$ is a subquotient of some $\mathrm{gr}^k_!(\Psi_{\xi,\mathbb{1}_v})$ with $k\geqslant 2$, then we must have $2\leqslant t\leqslant h_0+1$ and $\delta>0$.

In particular, from the previous important remark, we also deduce that

$$\chi'_v((t-1)/2-\delta)\equiv\chi_v(h_0/2) \mod \ell,$$

so that, by denoting $\lambda_1=\chi'_v(\varpi_v)$, we can write $\lambda_1 q_v^{h_0/2}=\lambda_0 q_v^{h_0/2+r}$ with $0<r<h_0+1$.

From Lemma 3.1.13, we can repeat the same argument with χ'_v in place of χ_v as it was announced.

References

[1] I. N. Bernstein & A. V. Zelevinsky – "Induced representations of reductive $\mathfrak{p}$-adic groups. I", *Ann. Sci. École Norm. Sup. (4)* **10** (1977), no. 4, p. 441–472.

[2] P. Boyer – "Local Ihara's lemma and applications", *Internat. Math. Res. Notices* (2021), article no. rnab298 (58 pages).

[3] P. Boyer – "Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples", *Invent. Math.* **177** (2009), no. 2, p. 239–280.

[4] ______, "Conjecture de monodromie-poids pour quelques variétés de Shimura unitaires", *Compositio Math.* **146** (2010), no. 2, p. 367–403.

[5] ______, "Réseaux d'induction des représentations elliptiques de Lubin-Tate", *J. Algebra* **336** (2011), p. 28–52.

[6] ______, "Filtrations de stratification de quelques variétés de Shimura simples", *Bull. Soc. math. France* **142** (2014), no. 4, p. 777–814.

[7] ______, "Sur la torsion dans la cohomologie des variétés de Shimura de Kottwitz-Harris-Taylor", *J. Inst. Math. Jussieu* **18** (2019), no. 3, p. 499–517.

[8] ______, "Groupe mirabolique, stratification de Newton raffinée et cohomologie des espaces de Lubin-Tate", *Bull. Soc. math. France* **148** (2020), no. 1, p. 1–23.

[9] ______, "p-stabilization in higher dimension", *J. Ramanujan Math. Soc.* **35** (2020), no. 2, p. 191–199.

[10] ______, "Ihara lemma and level raising in higher dimension", *J. Inst. Math. Jussieu* **21** (2022), no. 5, p. 1701–1726.

[11] ______, "La cohomologie des espaces de Lubin-Tate est libre", *Duke Math. J.* (to appear).

[12] A. Caraiani & P. Scholze – "On the generic part of the cohomology of compact unitary Shimura varieties", *Ann. of Math. (2)* **186** (2017), no. 3, p. 649–766.

[13] J.-F. Dat – "Un cas simple de correspondance de Jacquet-Langlands modulo ℓ", *Proc. London Math. Soc. (3)* **104** (2012), no. 4, p. 690–727.

[14] L. Fargues – "Filtration de monodromie et cycles évanescents formels", *Invent. Math.* **177** (2009), no. 2, p. 281–305.
[15] L. Fargues, A. Genestier & V. Lafforgue – *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in Math., vol. 262, Birkhäuser Verlag, Basel, 2008.
[16] M. Harris & R. Taylor – *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001.
[17] D. Juteau – "Decomposition numbers for perverse sheaves", *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 3, p. 1177–1229.
[18] T. Koshikawa – "Vanishing theorems for the mod p cohomology of some simple Shimura varieties", *Forum Math. Sigma* **8** (2020), article no. e38 (9 pages).
[19] P. Schneider & U. Stuhler – "The cohomology of p-adic symmetric spaces", *Invent. Math.* **105** (1991), no. 1, p. 47–122.
[20] P. Scholze – "On torsion in the cohomology of locally symmetric varieties", *Ann. of Math. (2)* **182** (2015), no. 3, p. 945–1066.
[21] ———, "On the p-adic cohomology of the Lubin-Tate tower", *Ann. Sci. École Norm. Sup. (4)* **51** (2018), no. 4, p. 811–863.
[22] R. Taylor & T. Yoshida – "Compatibility of local and global Langlands correspondences", *J. Amer. Math. Soc.* **20** (2007), no. 2, p. 467–493.
[23] M.-F. Vignéras – *Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$*, Progress in Math., vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996.
[24] A. V. Zelevinsky – "Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$", *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 2, p. 165–210.

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