Journal de l'École polytechnique
Mathématiques

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STABILIZATION OF THE DAMPED PLATE EQUATION
UNDER GENERAL BOUNDARY CONDITIONS

by Jérôme Le Rousseau & Emmanuel Wend-Benedo Zongo

Abstract. — We consider a damped plate equation on an open bounded subset of $\mathbb{R}^d$, or a smooth manifold, with boundary, along with general boundary operators fulfilling the Lopatinskiĭ-Šapiro condition. The damping term acts on an internal region without imposing any geometrical condition. We derive a resolvent estimate for the generator of the damped plate semigroup that yields a logarithmic decay of the energy of the solution to the plate equation. The resolvent estimate is a consequence of a Carleman inequality obtained for the bi-Laplace operator involving a spectral parameter under the considered boundary conditions. The derivation goes first through microlocal estimates, then local estimates, and finally a global estimate.

Résumé (Stabilisation de l'équation des plaques amorties sous des conditions au bord générales)
Nous considérons une équation des plaques amorties sur un ouvert borné régulier de $\mathbb{R}^d$, ou sur une variété lisse et compacte à bord, avec des opérateurs au bord généraux qui satisfont la condition de Lopatinskiĭ-Šapiro. Le terme d'amortissement agit sur une région interne et aucune condition géométrique n'est imposée. Nous démontrons une estimée de résolvante pour le générateur du semi-groupe associé qui implique une décroissance logarithmique de l'énergie de la solution de l'équation des plaques. Cette estimée de résolvante est conséquence d'une inégalité de Carleman obtenue pour le bi-laplacien muni d'un paramètre spectral et sous les conditions au bord considérées. L'obtention de cette inégalité passe tout d'abord par des estimations microlocales, puis locales et enfin une estimation globale.

Contents

1. Introduction

Let Ω be a bounded connected open subset in $\mathbb{R}^d$, or a smooth bounded connected d-dimensional manifold, with smooth boundary $\partial\Omega$, where we consider a damped plate equation

$$(1.1) \quad \begin{cases} \partial_t^2 y + \Delta^2 y + \alpha(x)\partial_t y = 0 & (t,x) \in \mathbb{R}_+ \times \Omega, \\ B_1 y_{|\mathbb{R}_+ \times \partial\Omega} = B_2 y_{|\mathbb{R}_+ \times \partial\Omega} = 0, \\ y_{|t=0} = y^0, \quad \partial_t y_{|t=0} = y^1, \end{cases}$$

where $\alpha \geqslant 0$ and where B_1 and B_2 denote two boundary differential operators. The damping property is provided by $+\alpha(x)\partial_t$ thus referred as the *damping term*. As introduced below, Δ^2 is the bi-Laplace operator, that is, the square of the Laplace operator. Here, it is associated with a smooth metric g to be introduced below; it is thus rather the bi-Laplace-Beltrami operator. This equation appears in models for the description of mechanical vibrations of thin domains. The two boundary operators are of order k_j, $j = 1, 2$ respectively, yet at most of order 3 in the direction normal to the boundary. They are chosen such that the two following properties are fulfilled:

(1) the Lopatinskiĭ-Šapiro boundary condition holds (this condition is fully described in what follows);

(2) along with the homogeneous boundary conditions given above the bi-Laplace operator is self-adjoint and nonnegative. This guarantees the conservation of the energy of the solution in the case of a damping free equation, that is, if $\alpha = 0$.

We are concerned with the decay of the energy of the solution in the case α is not identically zero. We shall prove that the damping term yields a stabilization property: the energy decays to zero as time t tends to infinity and we shall prove that the decay rate is at least logarithmic.

1.1. Stabilization and control of Schrödinger and plate equations. — In the case of the "hinged" boundary conditions

$$y_{|\mathbb{R}^+ \times \partial\Omega} = \Delta y_{|\mathbb{R}^+ \times \partial\Omega} = 0,$$

the plate equation can be written as the product of two Schrödinger equations since $\partial_t^2 y + \Delta^2 y = (-i\partial_t + \Delta)(i\partial_t + \Delta)$. As observed by Lebeau in [34] this allows one to transfer a control result obtained for the Schrödinger equation to the plates equations. In particular, Lebeau proved that controllability can be obtained for both equations if the control region fulfills the celebrated Rauch-Taylor condition, often coined GCC for geometrical control condition [39, 6]. The GCC expresses that all rays of geometrical optics reach the control region. Yet, the GCC condition is not a necessary condition as expressed by the result of Jaffard [23] on the controllability of the plate equation on a rectangle domain with an arbitrarily small control domain along with the "hinged" boundary conditions. Yet, the proof of [23] relies on the generalization of Ingham type inequalities in [25] and the very particular geometry that is considered.

For stabilization, GCC concerns the damping region, here given by the support of the function α. If GCC holds exponential stabilization holds. Note that in the same geometry as that of [23] an exponential stabilization result is proved in [38], using similar techniques. However, if no geometrical condition is imposed on the damping region, if a general geometry is considered, and if different boundary conditions are considered, one cannot expect an exponential decay rate. A logarithmic decay rate is quite natural if one has in mind the equivalent result obtained for the wave equation in the works of G. Lebeau [35] and G. Lebeau and L. Robbiano [37]. We also refer to [31, Chap. 6] and [32, Chap. 10 & 11] where the result of [35, 37] for the wave equation are reviewed and generalized in particular on the framework of general boundary conditions as those we consider here. See also the work of P. Cornilleau and L. Robbiano for a quite exotic boundary condition, namely the Zaremba condition.

Among the existing results available in the literature for plate type equations, many of them concern the "hinged" boundary conditions described above. In [38] where exponential stabilization is proved, the localized damping term involves the time derivative $\partial_t y$ as in (1.1). Interior nonlinear feedbacks can be used for exponential stabilization [41]. There, feedbacks are localized in a neighborhood of part of the boundary that fulfills multiplier-type conditions. A general analysis of nonlinear damping that includes the plate equation is provided in [2] under multiplier-type conditions. For "hinged" boundary conditions also, with a boundary damping term, we cite [4] where, on a square domain, a necessary and sufficient condition is provided for exponential stabilization.

Note that under "hinged" boundary conditions the bi-Laplace operator is precisely the square of the Dirichlet-Laplace operator. This makes its mathematical analysis much easier, in particular where using spectral properties, and this explains why this type of boundary conditions appears very frequently in the mathematical literature.

A more challenging type of boundary condition is the so-called "clamped" boundary conditions, that is, $u_{|\partial\Omega} = 0$ and $\partial_\nu u_{|\partial\Omega} = 0$, for which few results are available. We cite [1], where a general analysis of nonlinearly damped systems that includes the plate equation under multiplier-type conditions is provided. In [3], the analysis of discretized general nonlinearly damped system is also carried out, with the plate equation as an application. In [42], a nonlinear damping involving the p-*Laplacian* is used also under multiplier-type conditions. In [13], an exponential decay is obtained in the case of "clamped" boundary conditions, yet with a damping term of the Kelvin-Voigt type, that is of the form $\partial_t \Delta y$, that acts over the whole domain. In the case of the "clamped" boundary conditions, the logarithmic-type stabilization result we obtain here was proved in [33]. The present article thus stands as a generalization of the stabilization result of [33] if considering a whole class of boundary condition instead of specializing to a certain type. The present work contains in particular also the case of "hinged" boundary conditions.

1.2. **Method.** — Following the works of [33, 35, 37] we obtain a logarithmic decay rate for the energy of the solution to (1.1) by means of a resolvent estimate for the

generator of the semigroup associated with the damped plate equation (1.1). In [33] this was achieved by means of a Carleman estimate for the elliptic operator $D_s^4 + \Delta^2$ with an additional variable s in the case of the "clamped" boundary conditions. Extension of this strategy has however not been possible in the case of general boundary conditions. The proof of the resolvent estimate we seek is yet also possible by means of a Carleman estimate for the operator $P_\sigma = \Delta^2 - \sigma^4$ where σ is a spectral parameter for the generator of the semigroup and we found that this method of proof extends to general boundary conditions.

Our first goal is thus the derivation of the Carleman inequality for the operator P_σ near the boundary under the boundary conditions given by B_1 and B_2.

Then, from the Carleman estimate one deduces an observation inequality for the operator P_σ in the case of the prescribed boundary conditions. The resolvent estimate then follows from this observation inequality.

1.3. On Carleman estimates. — A Carleman estimate is a weighted a priori inequality for the solutions of a partial partial differential equation, where the weight is of exponential type. For instance, for a partial differential operator P away from the boundary, it takes the form

$$\|e^{\tau\varphi}u\|_{L^2(\Omega)} \leqslant C\|e^{\tau\varphi}Pu\|_{L^2(\Omega)},$$

for $u \in \mathscr{C}_c^\infty(\Omega)$ and $\tau \geqslant \tau_0$ for φ well chosen and some τ_0 chosen sufficiently large. The exponential weight function involves a parameter τ that can be taken as large as desired, making Carleman inequalities very powerful estimates. Additional terms on the left-hand side of the inequality can be obtained, including higher-order derivatives of the function u, depending of course of the order of the operator P itself. For a second-order elliptic operator such as the Laplace operator one has

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2(\Omega)} + \tau^{1/2}\|e^{\tau\varphi}Du\|_{L^2(\Omega)} + \tau^{-1/2}\sum_{|\beta|=2}\|e^{\tau\varphi}D^\beta u\|_{L^2(\Omega)}$$
$$\leqslant C\|e^{\tau\varphi}\Delta u\|_{L^2(\Omega)},$$

under the so-called sub-ellipticity condition; see [31, Chap. 3]. Note that the power of the large parameter τ adds to $3/2$ with the order of the derivative in each term on the left-hand side. In fact, in the calculus used to derive such estimates one power of τ is equivalent to a derivative of order one. Thus with this $3/2$ compared with the order two of the operator one says that one looses a half-derivative in the estimate.

This type of estimate was used for the first time by T. Carleman [12] to achieve uniqueness properties for the Cauchy problem of an elliptic operator. Later, A.-P. Calderón and L. Hörmander further developed Carleman's method [11, 18]. To this day, the method based on Carleman estimates remains essential to prove unique continuation properties; see for instance [43] for an overview. On such questions, more recent advances have been concerned with differential operators with singular potentials, starting with the contribution of D. Jerison and C. Kenig [24]. The reader is also referred to [40, 27, 28, 14, 29]. In more recent years, the field of applications of

Carleman estimates has gone beyond the original domain; they are also used in the study of:

– Inverse problems, where Carleman estimates are used to obtain stability estimates for the unknown sought quantity (for instance coefficient, source term) with respect to norms on measurements performed on the solution of the PDE, see for instance [10, 22, 30, 21]; Carleman estimates are also fundamental in the construction of complex geometrical optic solutions that lead to the resolution of inverse problems such as the Calderón problem with partial data [26, 15].

– Control theory for PDEs; Carleman estimates yield the null controllability of linear parabolic equations [36] and the null controllability of classes of semi-linear parabolic equations [17, 5, 16]. They can also be used to prove unique continuation properties, that in turn are crucial for the treatment of low frequencies for exact controllability results for hyperbolic equations as in [6].

For a function supported near a point at the boundary, in normal geodesic coordinates where Ω is locally given by $\{x_d > 0\}$ (see Section 1.4 below) the estimate can take the form

$$\sum_{|\beta|\leqslant 2} \tau^{3/2-|\beta|}\|e^{\tau\varphi}D^{\beta}u\|_{L^2(\Omega)} + \sum_{|\beta|\leqslant 1} \tau^{3/2-|\beta|}|e^{\tau\varphi}D^{\beta}u_{|x_d=0^+}|_{L^2(\Omega)}$$

$$\leqslant C\|e^{\tau\varphi}\Delta u\|_{L^2(\Omega)}.$$

This is the type of estimate we seek here for the operator P_σ, with some uniformity with respect to σ.

1.4. **Geometrical setting.** — On Ω we consider a Riemannian metric $g_x = (g_{ij}(x))$, with associated cometric $(g^{ij}(x)) = (g_x)^{-1}$. It stands as a bilinear form that act on vector fields,

$$g_x(u_x, v_x) = g_{ij}(x)u_x^i v_x^j, \quad u_x = u_x^i \partial_{x_i}, \ v_x = v_x^i \partial_{x_i}.$$

For $x \in \partial\Omega$ we denote by ν_x the unit outward pointing *normal vector* at x, unitary in the sense of the metric g, that is

$$g_x(\nu_x, \nu_x) = 1 \ \text{ and } \ g_x(\nu_x, u_x) = 0 \quad \forall u_x \in T_x\partial\Omega.$$

We denote by ∂_ν the associated derivative at the boundary, that is, $\partial_\nu f(x) = \nu_x(f)$. We also denote by n_x the unit outward pointing *conormal vector* at x, that is, $n_x = \nu_x^\flat$, that is, $(n_x)_i = g_{ij}\nu_x^j$.

Near a boundary point we shall often use normal geodesic coordinates where Ω is locally given by $\{x_d > 0\}$ and the metric g takes the form

$$g = dx^d \otimes dx^d + \sum_{1\leqslant i,j\leqslant d-1} g_{ij}dx^i \otimes dx^j.$$

Then, the vector field ν_x is locally given by $(0, \ldots, 0, -1)$. The same for the one form n_x.

Normal geodesic coordinates allow us to locally formulate boundary problems in a half-space geometry. We write

$$\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \mid x_d > 0\} \qquad \text{where } x = (x', x_d) \text{ with } x' \in \mathbb{R}^{d-1},\, x_d \in \mathbb{R}.$$

We shall denote its closure by $\overline{\mathbb{R}^d_+}$, that is, $\overline{\mathbb{R}^d_+} = \{x \in \mathbb{R}^d \mid x_d \geqslant 0\}$.

The Laplace-Beltrami operator is given by

$$(1.2) \qquad (\Delta_g f)(x) = (\det g_x)^{-1/2} \sum_{1 \leqslant i,j \leqslant d} \partial_{x_i}\big((\det g_x)^{1/2} g^{ij}(x) \partial_{x_j} f\big)(x).$$

in local coordinates. Its principal part is given by $\sum_{1 \leqslant i,j \leqslant d} g^{ij}(x) \partial_{x_i} \partial_{x_j}$ and its principal symbol by $\sum_{1 \leqslant i,j \leqslant d} g^{ij}(x) \xi_i \xi_j$.

The bi-Laplace operator is $P = \Delta_g^2$. In the main text of the article we shall write Δ, Δ^2 in place of Δ_g, Δ_g^2.

1.5. Main results

1.5.1. *Stabilization result.* —
Let $(\mathsf{P}_0, D(\mathsf{P}_0))$ be the unbounded operator on $L^2(\Omega)$ given by the domain

$$(1.3) \qquad D(\mathsf{P}_0) = \big\{u \in H^4(\Omega) \mid B_1 u_{|\partial\Omega} = B_2 u_{|\partial\Omega} = 0\big\},$$

given by $\mathsf{P}_0 u = \Delta^2 u$ for $u \in D(\mathsf{P}_0)$. As written above the two boundary differential operators are such that $(\mathsf{P}_0, D(\mathsf{P}_0))$ is self-adjoint and nonnegative.

Let $y(t)$ be a strong solution of the plate equation (1.1). A precise definition of strong solutions is given in Section 9.3. One has $y^0 \in D(\mathsf{P}_0)$ and $y^1 \in D(\mathsf{P}_0^{1/2})$. Its energy is defined as

$$\mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|^2_{L^2(\Omega)} + (\mathsf{P}_0 y(t), y(t))_{L^2(\Omega)}\big).$$

Theorem 1.1 (logarithmic stabilization for the damped plate equation)

There exists $C > 0$ such that for any such strong solution to the damped plate equation (1.1) *one has*

$$\mathcal{E}(y)(t) \leqslant \frac{C}{\big(\log(2+t)\big)^4}\big(\|\mathsf{P}_0 y^0 + \alpha y^1\|^2_{L^2(\Omega)} + \|\mathsf{P}_0^{1/2} y^1\|^2_{L^2(\Omega)}\big).$$

A more precise and more general statement is given in Theorem 10.3 in Section 10.2. As explained in Section 1.2 the above stabilization result will be proved thanks to a Carleman estimate that we present now.

1.5.2. *Carleman estimate.* —
We state the main Carleman estimate for the operator $P_\sigma = \Delta^2 - \sigma^4$ in normal geodesic coordinates as presented in Section 1.4. A point $x^0 \in \partial\Omega$ is considered and a weight function φ is assumed to be defined locally and such that

(1) $\partial_d \varphi \geqslant C > 0$ locally.

(2) $(\Delta \pm \sigma^2, \varphi)$ satisfies the sub-ellipticity condition of Definition 6.1 locally. This is a necessary and sufficient condition for a Carleman estimate to hold for a second-order operator $\Delta \pm \sigma^2$, regardless of boundary conditions [31, Chap. 3 and 4].

(3) $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geqslant \kappa_0 \sigma$, for some $\kappa_0 > 0$. This means that the Lopatinskiĭ-Šapiro condition holds after the conjugation of the operator P_σ and the boundary operators B_1 and B_2 by the weight function $\exp(\tau\varphi)$.

THEOREM 1.2 (Carleman estimate for $P_\sigma = \Delta^2 - \sigma^4$). — *Let $\kappa_0' > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$. Let φ be such that the properties above hold locally. Then, there exists a neighborhood W^0 of x^0, $C > 0$, $\tau_0 > 0$ such that*

$$(1.4) \quad \tau^{-1/2} \|e^{\tau\varphi} u\|_{4,\tau} + |\operatorname{tr}(e^{\tau\varphi} u)|_{3,1/2,\tau}$$
$$\leqslant C \Big(\|e^{\tau\varphi} P_\sigma u\|_+ + \sum_{j=1}^2 |e^{\tau\varphi} B_j v_{|x_d=0^+}|_{7/2-k_j,\tau} \Big),$$

for $\tau \geqslant \tau_0$, $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$, and $u \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The volume norm is given by

$$\|e^{\tau\varphi} u\|_{4,\tau} = \sum_{|\beta| \leqslant 4} \tau^{4-|\beta|} \|e^{\tau\varphi} D^\beta u\|_{L^2(\Omega)}.$$

The trace norm is given by

$$|\operatorname{tr}(e^{\tau\varphi} u)|_{3,1/2,\tau} = \sum_{0 \leqslant n \leqslant 3} |\partial_\nu^n (e^{\tau\varphi} u)_{|x_d=0^+}|_{7/2-n,\tau},$$

where the norm $|.|_{7/2-n,\tau}$ is the L^2-norm in $\mathbb{R}^{d-1}$ after applying the Fourier multiplier $(\tau^2 + |\xi'|^2)^{7/4-n/2}$. These norms are well described in Section 2.3.

Observe that the Carleman estimate of Theorem 1.2 exhibits a loss of a half-derivative. A more precise statement is given in Theorem 7.4 in Section 7.2.

REMARK 1.3. — The condition $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$ in Theorem 1.2 calls for a comment as it is not classical in the case of Carleman estimates, even in the presence of a spectral parameter. Usually one has only $\kappa_0\sigma \leqslant \tau$ and in applications similar to that we have in mind one chooses $\tau = \kappa_0\sigma$. The extended condition we make here thus does not appear as a potential limitation.

We postpone the technical explanation to Remark 7.2 below Proposition 7.1 where this condition appears first.

1.6. SOME OPEN QUESTIONS

1.6.1. *Boundary damping*. — Here, we have considered a damping that acts in the interior of the domain Ω. The study of boundary damping, as in [37] for the wave equation, is also of relevance. Yet we foresee that it requires to specify more the used boundary operators. This was not our goal as we wished to treat general boundary operators here.

1.6.2. *Spectral inequality*. — If the boundary operators B_1 and B_2 are well chosen, the bi-Laplace operator Δ^2 can be selfadjoint on $L^2(\Omega)$; see Section 9.1. Associated with the operator is then a Hilbert basis $(\phi_j)_{j \in \mathbb{N}}$ of $L^2(\Omega)$. In the case of "clamped" boundary condition the following spectral inequality was proved in [33].

Theorem 1.4 (Spectral inequality for the "clamped" bi-Laplace operator)
    Let $\varnothing$ be an open subset of Ω. There exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C e^{C\mu^{1/4}} \|u\|_{L^2(\varnothing)}, \qquad \mu > 0, \quad u \in \mathrm{Span}\{\phi_j \mid \mu_j \leqslant \mu\}.$$

The proof of this theorem is based on a Carleman inequality for the fourth-order elliptic operator $D_s^4 + \Delta^2$, that is, after the addition of a variable s. Extending this strategy to the type of boundary conditions treated here was not successful because it is not guaranteed that having the Lopatinskiĭ-Šapiro condition for Δ^2, B_1, and B_2 implies that the Lopatinskiĭ-Šapiro condition holds for $D_s^4 + \Delta^2$, B_1, and B_2. Yet, the Lopatinskiĭ-Šapiro condition is at the heart of the proof of our Carleman estimate. Proving a spectral estimate as in the above statement for the general boundary conditions considered here is an open question.

1.6.3. *Quantification of the unique continuation property*. — For a second-order operator like the Laplace operator Δ and a boundary operator B of order k (yet of order at most one in the normal direction) such that the Lopatinskiĭ-Šapiro condition holds one can derive the following inequality that locally quantifies the unique continuation property up to the boundary; see [31, Lem. 9.2].

Proposition 1.5. — *Let $x^0 \in \partial\Omega$ and V be a neighborhood of x^0 where the Lopatinskiĭ-Šapiro condition holds. There exist a neighborhood W of x^0, $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$, and $C > 0$, such that*

$$(1.5) \qquad \|u\|_{H^1(W)} \leqslant C \|u\|_{H^1(V)}^{1-\delta} \Big(\|\Delta u\|_{L^2(V)} + |Bu|_{H^{1-k}(V \cap \partial\Omega)} + \|u\|_{H^1(V_\varepsilon)} \Big)^\delta,$$

for $u \in H^2(V)$, with $V_\varepsilon = \{x \in V \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$.

This inequality can be obtained from a Carleman estimate as in Theorem 1.2 for the Laplace operator, yet with the large parameter τ allowed to be chosen as large as desired. This is exploited in an optimization procedure on the parameter τ in $[\tau_0, +\infty[$ for some τ_0. Note however that in the statement of Theorem 1.2 one has $\sigma \lesssim \tau \lesssim \sigma$. Thus, the optimization procedure cannot be carried out in $[\tau_0, +\infty[$. While having a result quantifying the unique continuation under Lopatinskiĭ-Šapiro-type conditions for the bi-Laplace operator similar to that of Proposition 1.5 can be expected, the Carleman estimate we obtain in the present article cannot be used, at least directly, for a proof as carried out in [31] or in former sources such as [36].

1.7. Outline. — This article is organized as follows. In Section 2 we recall some basic aspects of pseudo-differential operators with a large parameters $\tau > 0$ and some positivity inequality of Gårding type in particular for quadratic forms in a half-space or at the boundary. Associated with the large parameters are Sobolev like norms, also in a half-space or at the boundary.

In Section 3, the Lopatinskiĭ-Šapiro boundary condition is properly defined for an elliptic operator, we give examples focusing on the Laplace and bi-Laplace operator and we give a formulation in local normal geodesic coordinates that we shall mostly use throughout the article. For the bi-Laplace operator we provide a series of examples of boundary operators for which the Lopatinskiĭ-Šapiro boundary condition holds and moreover the resulting operator is symmetric. We also show that the algebraic conditions that characterize the Lopatinskiĭ-Šapiro condition is robust under perturbation. This last aspect is key in the understanding of how the Lopatinskiĭ-Šapiro condition get preserved under conjugation and the introduction of a spectral parameter. This is done in Section 4, where an analysis of the configuration of the roots of the conjugated bi-Laplace operator is performed. In Section 4.5 the Lopatinskiĭ-Šapiro condition for the conjugated operator is exploited to obtain a symbol positivity for a quadratic form to prepare for the derivation of a Carleman estimate.

In Section 5 we derive an estimation of the boundary traces. This is precisely where the Lopatinskiĭ-Šapiro condition is used. The result is first obtained microlocally and we then apply a patching procedure.

To obtain the Carleman estimate for the bi-Laplace operator with spectral parameter $\Delta^2 - \sigma^4$ in Section 6 we first derive microlocal estimates for the operators $\Delta \pm \sigma^2$. Imposing σ to be non-zero, in the sense that $\sigma \gtrsim \tau$, the previous estimates exhibits losses in different microlocal regions. Thus concatenating the two estimates one derives an estimate for $\Delta^2 - \sigma^4$ where losses do not accumulates. A local Carleman estimate with only a loss of a half-derivative is obtained. This is done in Section 7. With the traces estimation obtained in Section 5 one obtains the local Carleman estimate of Theorem 1.2.

For the application to stabilization we have in mind, in Section 8 we use a global weight function and derive a global version of the Carleman estimate for $\Delta^2 - \sigma^4$ on the whole Ω. This leads to an observability inequality.

In Section 9 we recall aspects of strong and weak solutions to the damped plate equation, in particular through a semigroup formulation. With the observability inequality obtained in Section 8 we derive in Section 10 a resolvent estimate for the generator of the plate semigroup that in turn implies the stabilization result of Theorem 1.1.

1.8. Some notation. — The canonical inner product in $\mathbb{C}^m$ is denoted by

$$(z, z')_{\mathbb{C}^m} = \sum_{k=0}^{m-1} z_k \overline{z'}_k, \quad \text{for } z = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m, \ z' = (z'_0, \dots, z'_{m-1}) \in \mathbb{C}^m.$$

The associated norm will be denoted $|z|^2_{\mathbb{C}^m} = \sum_{k=0}^{m-1} |z_k|^2$.

We shall use the notations $a \lesssim b$ for $a \leqslant Cb$ and $a \gtrsim b$ for $a \geqslant Cb$, with a constant $C > 0$ that may change from one line to another. We also write $a \asymp b$ to denote $a \lesssim b \lesssim a$.

For an open set U of $\mathbb{R}^d$ we set $U_+ = U \cap \mathbb{R}^d_+$ and

$$(1.6) \qquad \overline{\mathscr{C}}^\infty_c(U_+) = \{u = v_{|\mathbb{R}^d_+} \mid v \in \mathscr{C}^\infty_c(\mathbb{R}^d) \text{ and } \operatorname{supp}(v) \subset U\}.$$

We set $\overline{\mathscr{S}}(\mathbb{R}^d_+) = \{u_{|\mathbb{R}^d_+} \mid u \in \mathscr{S}(\mathbb{R}^d)\}$ with $\mathscr{S}(\mathbb{R}^d)$ the usual Schwartz space in $\mathbb{R}^d$:

$$u \in \mathscr{S}(\mathbb{R}^d) \iff u \in \mathscr{C}^\infty(\mathbb{R}^d) \text{ and } \forall \alpha, \beta \in \mathbb{N}^d \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta_x u(x)| < \infty.$$

We recall that the Poisson bracket of two smooth functions is given by

$$\{f, g\} = \sum_{j=1}^{d} (\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g).$$

Acknowledgements. — The authors wish to thank two anonymous reviewers that provided useful corrections and remarks that improved the readability of the article.

2. Pseudo-differential calculus

In a half-space geometry motivated by the normal geodesic coordinates introduced in Section 1.4 we shall use $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and we shall consider the operators $D_d = -i\partial_d$ and $D' = -i\partial'$, with $\partial' = (\partial_{x_1}, \ldots, \partial_{x_{d-1}})$.

2.1. Pseudo-differential operators with a large a parameter. — In this subsection we recall some notions on semi-classical pseudo-differential operators with large parameter $\tau \geqslant 1$. We denote by S^m_τ the space of smooth functions $a(x, \xi, \tau)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, with $\tau \geqslant 1$ as a large parameter, that satisfies the following : for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ and $m \in \mathbb{R}$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi, \tau)| \leqslant C_{\alpha,\beta} \lambda^{m-|\beta|}_\tau, \quad \text{where} \quad \lambda^2_\tau = \tau^2 + |\xi|^2,$$

for all $(x, \xi, \tau) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty)$. For $a \in S^m_\tau$, one defines the associated pseudo-differential operator of order m, denoted by $A = \operatorname{Op}(a)$,

$$Au(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi, \tau) \widehat{u}(\xi) d\xi, \qquad u \in \mathscr{S}(\mathbb{R}^d).$$

One says that a is the symbol of A. We denote Ψ^m_τ the set of pseudo-differential operators of order m. We shall denote by $\mathscr{D}^m_\tau$ the space of semi-classical differential operators, i.e, the case when the symbol $a(x, \xi, \tau)$ is a polynomial of order m in (ξ, τ).

2.2. Tangential pseudo-differential operators. — Here, we consider pseudo-differential operators that only act in the tangential direction x' with x_d behaving as a parameter. We shall denote by $S^m_{\top, \tau}$, the set of smooth functions $b(x, \xi', \tau)$ defined for $\tau \geqslant 1$ as a large parameter and satisfying the following: for all multi-indices $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{d-1}$ and $m \in \mathbb{R}$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_{\xi'} b(x, \xi', \tau)| \leqslant C_{\alpha,\beta} \lambda^{m-|\beta|}_{\top,\tau}, \quad \text{where} \quad \lambda^2_{\top,\tau} = \tau^2 + |\xi'|^2,$$

for all $(x, \xi', \tau) \in \mathbb{R}^d \times \mathbb{R}^{d-1} \times [1, \infty)$. For $b \in S^m_{\mathsf{T}, \tau}$, we define the associated tangential pseudo-differential operator $B = \mathrm{Op}_{\mathsf{T}}(b)$ of order m by

$$Bu(x) := \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{ix' \cdot \xi'} b(x, \xi', \tau) \widehat{u}(\xi', x_d) d\xi', \qquad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+).$$

We define $\Psi^m_{\mathsf{T}, \tau}$ as the set of tangential pseudo-differential operators of order m, and $\mathscr{D}^m_{\mathsf{T}, \tau}$ the set of tangential differential operators of order m. We also set

$$\Lambda^m_{\mathsf{T}, \tau} = \mathrm{Op}_{\mathsf{T}}(\lambda^m_{\mathsf{T}, \tau}).$$

Let $m \in \mathbb{N}$ and $m' \in \mathbb{R}$. If we consider a of the form

$$a(x, \xi, \tau) = \sum_{j=0}^m a_j(x, \xi', \tau) \xi_d^j, \quad a_j \in S^{m+m'-j}_{\mathsf{T}, \tau},$$

we define $\mathrm{Op}(a) = \sum_{j=0}^m \mathrm{Op}_{\mathsf{T}}(a_j) D_{x_d}^j$. We write $a \in S^{m, m'}_\tau$ and $\mathrm{Op}(a) \in \Psi^{m, m'}_\tau$.

2.3. FUNCTION NORMS. — For functions norms we use the notation $\|.\|$ for functions defined in the interior of the domain and $|.|$ for functions defined on the boundary. In that spirit, we shall use the notation

$$\|u\|_+ = \|u\|_{L^2(\mathbb{R}^d_+)}, \quad (u, \widetilde{u})_+ = (u, \widetilde{u})_{L^2(\mathbb{R}^d_+)},$$

for functions defined in $\mathbb{R}^d_+$ and

$$|w|_\partial = \|w\|_{L^2(\mathbb{R}^{d-1})}, \quad (w, \widetilde{w})_\partial = (w, \widetilde{w})_{L^2(\mathbb{R}^{d-1})},$$

for functions defined on $\{x_d = 0\}$, such as traces.

We introduce the following norms, for $m \in \mathbb{N}$ and $m' \in \mathbb{R}$,

$$\|u\|_{m, m', \tau} \asymp \sum_{j=0}^m \|\Lambda^{m+m'-j}_{\mathsf{T}, \tau} D_{x_d}^j u\|_+,$$

$$\|u\|_{m, \tau} = \|u\|_{m, 0, \tau} \asymp \sum_{j=0}^m \|\Lambda^{m-j}_{\mathsf{T}, \tau} D_{x_d}^j u\|_+,$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. One has

$$\|u\|_{m, \tau} \asymp \sum_{|\alpha| \leqslant m} \tau^{m - |\alpha|} \|D^\alpha u\|_+,$$

and in the case $m' \in \mathbb{N}$ one has

$$\|u\|_{m, m', \tau} \asymp \sum_{\substack{\alpha_d \leqslant m \\ |\alpha| \leqslant m + m'}} \tau^{m + m' - |\alpha|} \|D^\alpha u\|_+,$$

with $\alpha = (\alpha', \alpha_d) \in \mathbb{N}^d$.

The following argument will be used on numerous occasions: for $m \in \mathbb{N}$, $m', \ell \in \mathbb{R}$, with $\ell > 0$,

$$\|u\|_{m, m', \tau} \ll \|u\|_{m, m' + \ell, \tau}$$

if τ is chosen sufficiently large.

At the boundary $\{x_d = 0\}$ we define the following norms, for $m \in \mathbb{N}$ and $m' \in \mathbb{R}$,

$$| \operatorname{tr}(u)|^2_{m,m',\tau} = \sum_{j=0}^{m} |\Lambda_{\mathsf{T},\tau}^{m+m'-j} D_{x_d}^j u_{|x_d=0^+}|^2_\partial, \qquad u \in \overline{\mathscr{S}}(\mathbb{R}_+^d).$$

2.4. **Differential quadratic forms.** — A differential quadratic form acts on a function and involves differentiations of various degrees of the function. One can associate to these forms a symbol and positivity inequality can be obtained in the form of a Gårding inequality. Such forms appear in proofs of Carleman estimates in the seminal work of Hörmander [19, §8.2].

Here differential quadratic forms are defined either in a half-space or at the boundary. The results we present here without proof can be found in [8] and [32].

2.4.1. *Differential quadratic forms in a half-space*

Definition 2.1 (interior differential quadratic form). — Let $u \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$. We say that

$$(2.1) \qquad Q(u) = \sum_{s=1}^{N} (A^s u, B^s u)_+, \qquad A^s = \operatorname{Op}(a^s), \ B^s = \operatorname{Op}(b^s),$$

is an interior differential quadratic form of type (m, r) with smooth coefficients if, for each $s = 1, \ldots N$, we have $a^s(\varrho) \in S_\tau^{m,r'}$ and $b^s(\varrho) \in S_\tau^{m,r''}$, with $r' + r'' = 2r$, $\varrho = (x, \xi, \tau)$.

The principal symbol of the quadratic form Q is defined as the class of

$$(2.2) \qquad q(\varrho) = \sum_{s=1}^{N} a^s(\varrho)\overline{b^s}(\varrho)$$

in $S_\tau^{2m,2r}/S_\tau^{2m,2r-1}$.

A result we shall use is the following microlocal Gårding inequality.

Proposition 2.2 (microlocal Gårding inequality). — *Let K be a compact set of $\overline{\mathbb{R}_+^d}$ and let $\mathscr{U}$ be an conic open set of $\overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ contained in $K \times \mathbb{R}^{d-1} \times \mathbb{R}_+$. Let also $\chi \in S_{\mathsf{T},\tau}^0$ be homogeneous of degree 0 and such that $\operatorname{supp}(\chi) \subset \mathscr{U}$. Let Q be an interior differential quadratic form of type (m, r) with homogeneous principal symbol $q \in S_\tau^{2m,2r}$ satisfying, for some $C_0 > 0$ and $\tau_0 > 0$,*

$$\operatorname{Re} q(\varrho) \geqslant C_0 \lambda_\tau^{2m} \lambda_{\mathsf{T},\tau}^{2r},$$

for $\tau \geqslant \tau_0$, $\varrho = (\varrho', \xi_d)$, $\varrho' = (x, \xi', \tau) \in \mathscr{U}$, and $\xi_d \in \mathbb{R}$.

For $0 < C_1 < C_0$ and $N \in \mathbb{N}$ there exist τ_*, $C > 0$, and $C_N > 0$ such that

$$\operatorname{Re} Q(\operatorname{Op}_\mathsf{T}(\chi)u) \geqslant C_1 \|\operatorname{Op}_\mathsf{T}(\chi)u\|^2_{m,r,\tau} - C| \operatorname{tr}(\operatorname{Op}_\mathsf{T}(\chi)u)|^2_{m-1,r+1/2,\tau} - C_N \|u\|^2_{m,-N,\tau},$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$ and $\tau \geqslant \tau_*$.

We refer to [8, Prop. 3.5] and [32, Th. 6.17] for a proof. We also state a local version of the result that follows from Proposition 2.2.

PROPOSITION 2.3 (Gårding inequality). — *Let U_0 be a bounded open subset of $\overline{\mathbb{R}^d_+}$ and let Q be an interior differential quadratic form of type (m,r) with homogeneous principal symbol $q \in S_\tau^{2m,2r}$ satisfying, for some $C_0 > 0$ and $\tau_0 > 0$,*

$$\operatorname{Re} q(\varrho) \geqslant C_0 \lambda_\tau^{2m} \lambda_{\mathsf{T},\tau}^{2r},$$

for $\tau \geqslant \tau_0$, $\varrho = (\varrho', \xi_d)$, $\varrho' = (x, \xi', \tau) \in U_0 \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, and $\xi_d \in \mathbb{R}$.
   For $0 < C_1 < C_0$ there exist τ_*, $C > 0$ such that

$$\operatorname{Re} Q(u) \geqslant C_1 \|u\|_{m,r,\tau}^2 - C |\operatorname{tr}(u)|_{m-1,r+1/2,\tau}^2,$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geqslant \tau_*$.

2.4.2. *Boundary differential quadratic forms*

DEFINITION 2.4. — Let $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. We say that

$$Q(u) = \sum_{s=1}^N (A^s u_{|x_d=0^+}, B^s u_{|x_d=0^+})_\partial, \quad A^s = a^s(x, D, \tau), \ B^s = b^s(x, D, \tau),$$

is a boundary differential quadratic form of type $(m-1, r)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \dots N$, we have $a^s(\varrho) \in S_\tau^{m-1,r'}(\overline{\mathbb{R}^d_+} \times \mathbb{R}^d)$, $b^s(\varrho) \in S_\tau^{m-1,r''}(\overline{\mathbb{R}^d_+} \times \mathbb{R}^d)$ with $r' + r'' = 2r$, $\varrho = (\varrho', \xi_d)$ with $\varrho' = (x, \xi', \tau)$. The symbol of the boundary differential quadratic form Q is defined by

$$q(\varrho', \xi_d, \widetilde{\xi}_d) = \sum_{s=1}^N a^s(\varrho', \xi_d) \overline{b^s}(\varrho', \widetilde{\xi}_d).$$

   For $\boldsymbol{z} = (z_0, \dots, z_{\ell-1}) \in \mathbb{C}^\ell$ and $a(\varrho) \in S_\tau^{\ell-1,t}$, of the form $a(\varrho', \xi_d) = \sum_{j=0}^{\ell-1} a_j(\varrho') \xi_d^j$ with $a_j(\varrho') \in S_{\mathsf{T},\tau}^{\ell-1+t-j}$ we set

$$(2.3) \qquad \Sigma_a(\varrho', \boldsymbol{z}) = \sum_{j=0}^{\ell-1} a_j(\varrho') z_j.$$

From the boundary differential quadratic form Q we introduce the following bilinear symbol $\Sigma_Q : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$

$$(2.4) \qquad \Sigma_Q(\varrho', \boldsymbol{z}, \boldsymbol{z}') = \sum_{s=1}^N \Sigma_{a^s}(\varrho', \boldsymbol{z}) \overline{\Sigma_{b^s}(\varrho', \overline{\boldsymbol{z}}')}, \quad \boldsymbol{z}, \boldsymbol{z}' \in \mathbb{C}^m.$$

   We let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$.

DEFINITION 2.5

   Let Q be a boundary differential quadratic form of type $(m-1, r)$ with homogeneous principal symbol and associated with the bilinear symbol $\Sigma_Q(\varrho', \boldsymbol{z}, \boldsymbol{z}')$. We say that Q is positive definite in $\mathscr{W}$ if there exist $C > 0$ and $R > 0$ such that

$$\operatorname{Re} \Sigma_Q(\varrho'', x_d = 0^+, \boldsymbol{z}, \boldsymbol{z}) \geqslant C \sum_{j=0}^{m-1} \lambda_{\mathsf{T},\tau}^{2(m-1-j+r)} |z_j|^2,$$

for $\varrho'' = (x', \xi', \tau) \in \mathscr{W}$, with $|(\xi', \tau)| \geqslant R$, and $\boldsymbol{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$.

We have the following Gårding inequality.

PROPOSITION 2.6. — *Let Q be a boundary differential quadratic form of type $(m-1, r)$, positive definite in $\mathscr{W}$, an open conic set in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, with bilinear symbol $\Sigma_Q(\varrho', \boldsymbol{z}, \boldsymbol{z}')$. Let $\chi \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, with $\operatorname{supp}(\chi_{|x_d=0^+}) \subset \mathscr{W}$ and let $N \in \mathbb{N}$. Then there exist $\tau_* \geqslant 1$, $C > 0$, $C_N > 0$ such that*

$$\operatorname{Re} Q(\operatorname{Op}_{\mathsf{T}}(\chi)u) \geqslant C |\operatorname{tr}(\operatorname{Op}_{\mathsf{T}}(\chi)u)|^2_{m-1,r,\tau} - C_N |\operatorname{tr}(u)|^2_{m-1,-N,\tau},$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geqslant \tau_*$.

2.5. SYMBOLS AND OPERATORS WITH AN ADDITIONAL LARGE PARAMETER. — In this article, we shall use operators with a symbol that depends on an additional large parameter σ, say $a(x, \xi, \tau, \sigma)$. They will satisfy estimate of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \tau, \sigma)| \leqslant C_{\alpha,\beta}(\tau^2 + |\xi|^2 + \sigma^2)^{(m-|\beta|)/2}.$$

We observe that if $\tau \gtrsim \sigma$ one has

$$\lambda_\tau^2 \leqslant \tau^2 + |\xi|^2 + \sigma^2 \lesssim \lambda_\tau^2.$$

Thus, as far as pseudo-differential calculus is concerned it is as if $a \in S^m_\tau$ and this property will be exploited in what follows.

Similarly if $a = a(x, \xi', \tau, \sigma)$ fulfills a tangential-type estimate of the form

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi', \tau, \sigma)| \leqslant C_{\alpha,\beta}(\tau^2 + |\xi'|^2 + \sigma^2)^{(m-|\beta|)/2},$$

if one has $\tau \gtrsim \sigma$ one will be able to apply techniques adapted to symbols in $S^m_{\mathsf{T},\tau}$ and associated operators, like for instance the results on differential quadratic forms listed in Section 2.4.

3. Lopatinskiĭ-Šapiro boundary conditions for elliptic operator

Let P be an elliptic differential operator of order $2k$ on Ω, ($k \geqslant 1$), with principal symbol $p(x, \omega)$ for $(x, \omega) \in T^*\Omega$. One defines the following polynomial in z,

$$\widetilde{p}(x, \omega', z) = p(x, \omega' - z n_x),$$

for $x \in \partial\Omega$, $\omega' \in T^*_x \partial\Omega$, $z \in \mathbb{R}$ and where n_x denotes the outward unit pointing conormal vector at x (see Section 1.4). Here x and ω' are considered to act as parameters. We denote by $\rho_j(x, \omega')$, $1 \leqslant j \leqslant 2k$ the complex roots of $z \mapsto \widetilde{p}(x, \omega', z)$. One sets

$$\widetilde{p}^+(x, \omega', z) = \prod_{\operatorname{Im} \rho_j(x,\omega') \geqslant 0} (z - \rho_j(x, \omega')).$$

Given boundary operators $B_1, \dots, B_k$ in a neighborhood of $\partial\Omega$, with principal symbols $b_j(x, \omega)$, $j = 1, \dots, k$, one also sets $\widetilde{b}_j(x, \omega', z) = b_j(x, \omega' - z n_x)$.

DEFINITION 3.1 (Lopatinskiĭ-Šapiro boundary condition). — Let $(x, \omega') \in T^*\partial\Omega$ with $\omega' \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for $(P, B_1, \dots, B_k)$ at (x, ω') if for any polynomial $f(z)$ with complex coefficients, there exists $c_1, \dots, c_k \in \mathbb{C}$ and a polynomial $g(z)$ with complex coefficients such that, for all $z \in \mathbb{C}$,

$$f(z) = \sum_{1 \leqslant j \leqslant k} c_j \widetilde{b}_j(x, \omega', z) + g(z) \widetilde{p}^+(x, \omega', z).$$

We say that the Lopatinskiĭ-Šapiro condition holds for $(P, B_1, \ldots, B_k)$ at $x \in \partial\Omega$ if it holds at (x, ω') for all $\omega' \in T_x^* \partial\Omega$ with $\omega' \neq 0$.

The Lopatinskiĭ-Šapiro boundary condition is of great importance in the analysis of elliptic equations. In fact, the Lopatinskiĭ-Šapiro condition is *equivalent* to having the operator

$$L : H^{m+2k}(\Omega) \longrightarrow H^m(\Omega) \oplus H^{m+2k-\ell_1-1/2}(\partial\Omega) \oplus \cdots \oplus H^{m+2k-\ell_k-1/2}(\partial\Omega)$$

$$u \longmapsto (Pu, B_1 u_{|\partial\Omega}, \ldots, B_k u_{|\partial\Omega}),$$

to be Fredholm, if $m \geqslant 0$ and if each operator B_j is of order ℓ_j. This is done in great generality in [20, Chap. 20]. If P is the Laplace operator this is presented in great details in [32, Chap. 3].

Roughly speaking the Lopatinskiĭ-Šapiro condition states that modes exponentially small in the interior of the domain Ω (as frequency increases) yet not small at the boundary cannot be estimated without the use of a proper boundary operator. No boundary operator is needed for mode that do not have this asymptotic behavior. This is explained in [32, §2.1.1].

3.1. Some examples. — The Lopatinskiĭ-Šapiro condition holds in the following cases

– $P = -\Delta$ on Ω, with the Dirichlet boundary condition, $Bu_{|\partial\Omega} = u_{|\partial\Omega}$.

– $P = \Delta^2$ on Ω, along with the so-called clamped boundary conditions, i.e, $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_\nu u_{|\partial\Omega}$, where ν is the normal outward pointing unit vector to $\partial\Omega$; see Section 1.4.

– $P = \Delta^2$ on Ω, along with the so-called hinged boundary conditions, i.e, $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \Delta u_{|\partial\Omega}$.

3.2. Case of the bi-Laplace operator. — With $P = \Delta^2$ on Ω, along with the general boundary operators B_1 and B_2 of orders k_1 and k_2 respectively, we give a matrix criterion of the Lopatinskiĭ-Šapiro condition. The general boundary operators B_1 and B_2 are then given by

$$B_\ell(x, D) = \sum_{0 \leqslant j \leqslant \min(3, k_\ell)} B_\ell^{k_\ell - j}(x, D')(i\partial_\nu)^j, \qquad \ell = 1, 2,$$

with $B_\ell^{k_\ell - j}(x, D')$ differential operators acting in the tangential variables. We denote by $b_1(x, \omega)$ and $b_2(x, \omega)$ the principal symbols of B_1 and B_2 respectively. For $(x, \omega') \in T^* \partial\Omega$, we set

$$\widetilde{b}_\ell(x, \omega', z) = \sum_{0 \leqslant j \leqslant \min(3, k_\ell)} b_\ell^{k_\ell - j}(x, \omega') z^j, \qquad \ell = 1, 2.$$

We recall that the principal symbol of P is given by $p(x, \omega) = |\omega|_g^4$. One thus has

$$\widetilde{p}(x, \omega', z) = p(x, \omega' - z n_x) = \left(z^2 + |\omega'|_g^2\right)^2.$$

Therefore $\widetilde{p}(x, \omega', z) = (z - i|\omega'|_g)^2 (z + i|\omega'|_g)^2$. According to the above definition we set $\widetilde{p}^+(x, \omega', z) = (z - i|\omega'|_g)^2$. Thus, the Lopatinskiĭ-Šapiro condition holds at (x, ω') with $\omega' \neq 0$ if and only if for any function $f(z)$ the complex number $i|\omega'|_g$ is a root

of the polynomial $z \mapsto f(z) - c_1 \widetilde{b}_1(x, \omega', z) - c_2 \widetilde{b}_2(x, \omega', z)$ and its derivative for some $c_1, c_2 \in \mathbb{C}$. This leads to the following determinant condition.

Lemma 3.2. — *Let $P = \Delta^2$ on Ω, B_1 and B_2 be two boundary operators. If $x \in \partial\Omega$, $\omega' \in T_x^* \partial\Omega$, with $\omega' \neq 0$, the Lopatinskiĭ-Šapiro condition holds at (x, ω') if and only if*

$$(3.1) \qquad \det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x, \omega', z = i|\omega'|_g) \neq 0.$$

Remark 3.3. — With the determinant condition and homogeneity, we note that if the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2) at (x, ω') it also holds in a conic neighborhood of (x, ω') by continuity. If it holds at $x \in \Omega$, it also holds in a neighborhood of x.

3.3. **Formulation in normal geodesic coordinates.** — Near a boundary point $x \in \partial\Omega$, we shall use normal geodesic coordinates. These coordinates are recalled in Section 1.4. Then the principal symbols of Δ and Δ^2 are given by $\xi_d^2 + r(x, \xi')$ and $(\xi_d^2 + r(x, \xi'))^2$ respectively, where $r(x, \xi')$ is the principal symbol of a tangential differential elliptic operator $R(x, D')$ of order 2, with

$$r(x, \xi') = \sum_{1 \leqslant i,j \leqslant d-1} g^{ij}(x) \xi_i' \xi_j' \quad \text{and} \quad r(x, \xi') \geqslant C|\xi'|^2.$$

Here g^{ij} is the inverse of the metric g_{ij}. Below, we shall often write $|\xi'|_x^2 = r(x, \xi')$ and we shall also write $|\xi|_x^2 = \xi_d^2 + r(x, \xi')$, for $\xi = (\xi', \xi_d)$.

If $b_1(x, \xi)$ and $b_2(x, \xi)$ are the principal symbols of the boundary operators B_1 and B_2 in the normal geodesic coordinates then the Lopatinskiĭ-Šapiro condition for (P, B_1, B_2) with $P = \Delta^2$ at (x, ξ') reads

$$\det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} (x, \xi', \xi_d = i|\xi'|_x) \neq 0,$$

if $|\xi'|_x \neq 0$ according to Lemma 3.2. If the Lopatinskiĭ-Šapiro condition holds at some x^0, because of homogeneity, there exists $C_0 > 0$ such that

$$(3.2) \qquad \left| \det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} \right| (x^0, \xi', i|\xi'|_x) \geqslant C_0 |\xi'|_x^{k_1+k_2-1}, \qquad \xi' \in \mathbb{R}^{d-1}.$$

3.4. **Stability of the Lopatinskiĭ-Šapiro condition.** — To prepare for the study of how the Lopatinskiĭ-Šapiro condition behaves under conjugation with Carleman exponential weight and the addition of a spectral parameter, we introduce some perturbations in the formulation of the Lopatinskiĭ-Šapiro condition for (P, B_1, B_2) as written in (3.2).

Lemma 3.4. — *Let V^0 be a compact set of $\partial\Omega$ such that the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2) at every point x of V^0. There exist $C_1 > 0$ and $\varepsilon > 0$ such*

that

$$(3.3) \qquad \left| \det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} \right| (x, \xi' + \zeta', \xi_d = i|\xi'|_x + \delta) \geqslant C_1 |\xi'|_x^{k_1 + k_2 - 1},$$

for $x \in V^0$, $\xi' \in \mathbb{R}^{d-1}$, $\zeta' \in \mathbb{C}^{d-1}$, and $\delta \in \mathbb{C}$, if $|\zeta'| + |\delta| \leqslant \varepsilon |\xi'|_x$. Moreover one has

$$(3.4) \quad \left| \det \begin{pmatrix} b_1(x, \xi' + \zeta', \xi_d = i|\xi'|_x + \delta) \ b_2(x, \xi' + \zeta', \xi_d = i|\xi'|_x + \delta) \\ b_1(x, \xi' + \zeta', \xi_d = i|\xi'|_x + \widetilde{\delta}) \ b_2(x, \xi' + \zeta', \xi_d = i|\xi'|_x + \widetilde{\delta}) \end{pmatrix} \right|$$
$$\geqslant C_1 |\delta - \widetilde{\delta}| \, |\xi'|_x^{k_1 + k_2 - 1},$$

for $x \in V^0$, $\xi' \in \mathbb{R}^{d-1}$, $\zeta' \in \mathbb{C}^{d-1}$, and $\delta, \widetilde{\delta} \in \mathbb{C}$, if $|\zeta'| + |\delta| + |\widetilde{\delta}| \leqslant \varepsilon |\xi'|_x$.

Proof. — From (3.2), since V^0 is compact having the Lopatinskiĭ-Šapiro condition holding at every point x of V^0 means there exists $C_0 > 0$ such that

$$(3.5) \qquad \left| \det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} \right| (x, \xi', i|\xi'|_x) \geqslant C_0 |\xi'|_x^{k_1 + k_2 - 1}, \quad x \in V^0, \xi' \in \mathbb{R}^{d-1}.$$

The first part is a consequence of the mean value theorem, homogeneity and (3.5) with say $C_1 = C_0/2$.

For the second part it is sufficient to assume that $\delta \neq \widetilde{\delta}$ since the result is obvious otherwise. For $j = 1, 2$ one writes the Taylor formula

$$b_j(x, \xi' + \zeta', i|\xi'|_x + \widetilde{\delta})$$
$$= b_j(x, \xi' + \zeta', i|\xi'|_x + \delta) + (\widetilde{\delta} - \delta)\partial_{\xi_d} b_j(x, \xi' + \zeta', i|\xi'|_x + \delta)$$
$$+ (\widetilde{\delta} - \delta)^2 \int_0^1 (1 - s)\partial_{\xi_d}^2 b_j(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \, ds,$$

with $\delta_s = (1 - s)\delta + s\widetilde{\delta}$, yielding

$$\frac{1}{\widetilde{\delta} - \delta} \det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) \ b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ b_1(x, \xi' + \zeta', i|\xi'|_x + \widetilde{\delta}) \ b_2(x, \xi' + \zeta', i|\xi'|_x + \widetilde{\delta}) \end{pmatrix}$$
$$= \det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} (x, \xi' + \zeta', i|\xi'|_x + \delta)$$
$$+ (\widetilde{\delta} - \delta) \int_0^1 (1 - s) f(x, \xi', \zeta', \delta, s) \, ds,$$

with

$$f(x, \xi', \zeta', \delta, s) = \det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) & b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ \partial_{\xi_d}^2 b_1(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \ \partial_{\xi_d}^2 b_2(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \end{pmatrix}.$$

With homogeneity, if $|\zeta'| + |\delta| + |\widetilde{\delta}| \lesssim |\xi'|_x$ one finds

$$\left| \det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) & b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ \partial_{\xi_d}^2 b_1(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \ \partial_{\xi_d}^2 b_2(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \end{pmatrix} \right| \lesssim |\xi'|_x^{k_1 + k_2 - 2},$$

Thus with $|\delta - \widetilde{\delta}| \leqslant \varepsilon |\xi'|_x$, for $\varepsilon > 0$ chosen sufficiently small, using the first part of the lemma one obtains the second result. $\qquad\square$

3.5. Examples of boundary operators yielding symmetry. — We give some examples of pairs of boundary operators B_1, B_2 that

  (1) fulfill the Lopatinskiĭ-Šapiro condition,
  (2) yield symmetry for the bi-Laplace operator $P = \Delta^2$, that is,

$$(Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)}$$

for $u, v \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = B_j v_{|\partial\Omega} = 0$, $j = 1, 2$.

We first recall that following Green formula

$$(3.6) \quad (\Delta u, v)_{L^2(\Omega)} = (u, \Delta v)_{L^2(\Omega)} + (\partial_n u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)},$$

which applied twice gives $(Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)} + T(u, v)$ with

$$(3.7) \quad T(u, v) = (\partial_n \Delta u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (\Delta u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)}$$
$$+ (\partial_n u_{|\partial\Omega}, \Delta v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n \Delta v_{|\partial\Omega})_{L^2(\partial\Omega)}.$$

Using normal geodesic coordinates in a neighborhood of the whole boundary $\partial\Omega$ allows one to write $\Delta = \partial_n^2 + \Delta'$ where Δ' is the resulting Laplace operator on the boundary, that is, associated with the trace of the metric on $\partial\Omega$. Since Δ' is selfadjoint on $\partial\Omega$ this allows one to write formula (3.7) in the alternative forms

$$(3.8) \quad T(u, v) = (\partial_n^3 u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - ((\partial_n^2 + 2\Delta')u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)}$$
$$+ (\partial_n u_{|\partial\Omega}, (\partial_n^2 + 2\Delta')v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n^3 v_{|\partial\Omega})_{L^2(\partial\Omega)},$$

or

$$(3.9) \quad T(u, v) = ((\partial_n^3 + 2\Delta'\partial_n)u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (\partial_n^2 u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)}$$
$$+ (\partial_n u_{|\partial\Omega}, \partial_n^2 v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, (\partial_n^3 + 2\Delta'\partial_n)v_{|\partial\Omega})_{L^2(\partial\Omega)}.$$

We start our list of examples with the most basics ones.

Example 3.5 (Hinged boundary conditions). — This type of conditions refers to $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \Delta u_{|\partial\Omega}$. With (3.7) one finds $T(u, v) = 0$ in the case of homogeneous conditions, hence symmetry.

Note that the hinged boundary conditions are equivalent to having $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n^2 u_{|\partial\Omega}$. With the notation of Section 3 this gives $\widetilde{b}_1(x, \omega', z) = 1$ and $\widetilde{b}_2(x, \omega', z) = (-iz)^2 = -z^2$. It follows that

$$\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix}(x, \omega', z = i|\omega'|_g) = \det \begin{pmatrix} 1 & |\omega'|_g^2 \\ 0 & -2i|\omega'|_g \end{pmatrix} = -2i|\omega'|_g \neq 0$$

if $\omega' \neq 0$ and thus the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

With the hinged boundary conditions observe that the bi-Laplace operator is precisely the square of the Dirichlet-Laplace operator. This makes its analysis quite simple and this explains why this type of boundary condition is often chosen in the

mathematical literature. Observe that symmetry is then obvious without invoking the above formulas.

Example 3.6 (Clamped boundary conditions). — This type of conditions refers to $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$. With (3.8) one finds $T(u,v) = 0$ in the case of homogeneous conditions, hence symmetry. With the notation of Section 3 this gives $\widetilde{b}_1(x,\omega',z) = 1$ and $\widetilde{b}_2(x,\omega',z) = -iz$. It follows that

$$\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x,\omega', z = i|\omega'|_g) = \det \begin{pmatrix} 1 & |\omega'|_g \\ 0 & -i \end{pmatrix} = -i \neq 0.$$

Thus the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

Note that with the clamped boundary conditions the bi-Laplace operator cannot be seen as the square of the Laplace operator with some well chosen boundary condition as opposed to the case of the hinged boundary conditions displayed above.

Examples 3.7 (More examples)

(1) Take $B_1 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n \Delta u_{|\partial\Omega}$. With these boundary conditions the bi-Laplace operator is precisely the square of the Neumann-Laplace operator. The symmetry property is immediate and so is the Lopatinskiĭ-Šapiro condition.

(2) Take $B_1 u_{|\partial\Omega} = (\partial_n^2 + 2\Delta') u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n^3 u_{|\partial\Omega}$. With (3.8) one finds $T(u,v) = 0$ in the case of homogeneous conditions, hence symmetry.

We have $\widetilde{b}_1(x,\omega',z) = -z^2 - 2|\omega'|_g^2$ and $\widetilde{b}_2(x,\omega',z) = iz^3$ and

$$\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x,\omega', z = i|\omega'|_g) = \det \begin{pmatrix} -|\omega'|_g^2 & |\omega'|_g^3 \\ -2i|\omega'|_g & -3i|\omega'|_g^2 \end{pmatrix}$$
$$= 5i|\omega'|_g^4 \neq 0$$

if $\omega' \neq 0$ and thus the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

(3) Take $B_1 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^3 + A') u_{|\partial\Omega}$, with A' a symmetric differential operator of order less than or equal to three on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $a'(x,\omega') \neq 2|\omega'|_g^3$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq 2$ for $|\omega'|_g = 1$.

With (3.8) one finds

$$T(u,v) = (-A'u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} + (u_{|\partial\Omega}, A'v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\widetilde{b}_1(x,\omega',z) = -iz$ and $\widetilde{b}_2(x,\omega',z) = iz^3 + a'(x,\omega')$ with a' the principal symbol of A'.

$$\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x,\omega', z = i|\omega'|_g) = \det \begin{pmatrix} |\omega'|_g & |\omega'|_g^3 + a'(x,\omega') \\ -i & -3i|\omega'|_g^2 \end{pmatrix}$$
$$= i\big(a'(x,\omega') - 2|\omega'|_g^3\big) \neq 0,$$

if $\omega' \neq 0$ since $a'(x,\omega') \neq 2|\omega'|_g^3$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

(4) Take $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^2 + A'\partial_n)u_{|\partial\Omega}$ with A' a symmetric differential operator of order less than or equal to one on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $a'(x,\omega') \neq -2|\omega'|_g$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq -2$ for $|\omega'|_g = 1$. This is a refinement of the case of hinged boundary conditions given in Example 3.5 above.

With (3.8) one finds

$$T(u,v) = (A'\partial_n u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} + (\partial_n u_{|\partial\Omega}, -A'\partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\widetilde{b}_1(x,\omega',z) = 1$ and $\widetilde{b}_2(x,\omega',z) = -z^2 - iza'(x,\omega')$ with a' the principal symbol of A'.

$$
\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x, \omega', z = i|\omega'|_g) = \det \begin{pmatrix} 1 & |\omega'|_g^2 + |\omega'|_g a'(x,\omega') \\ 0 & -2i|\omega'|_g - ia'(x,\omega') \end{pmatrix}
$$
$$
= -i\big(a'(x,\omega') + 2|\omega'|_g\big) \neq 0,
$$

if $\omega' \neq 0$ since $a'(x,\omega') \neq -2|\omega'|_g$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

(5) Take $B_1 u_{|\partial\Omega} = (\partial_n^2 + A'\partial_n)u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^3 + 2\partial_n\Delta')u_{|\partial\Omega}$, with A' a symmetric differential operator of order less than or equal to one on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $2a'(x,\omega') \neq -3|\omega'|_g$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq -3/2$ for $|\omega'|_g = 1$. With (3.9) one finds

$$T(u,v) = (A'\partial_n u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} + (\partial_n u_{|\partial\Omega}, -A'\partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\widetilde{b}_1(x,\omega',z) = -z^2 - iza'(x,\omega')$ and $\widetilde{b}_2(x,\omega',z) = iz^3 + 2iz|\omega'|_g^2$ and

$$
\det \begin{pmatrix} \widetilde{b}_1 & \widetilde{b}_2 \\ \partial_z \widetilde{b}_1 & \partial_z \widetilde{b}_2 \end{pmatrix} (x, \omega', z = i|\omega'|_g) = \det \begin{pmatrix} |\omega'|_g^2 + |\omega'|_g a'(x,\omega') & -|\omega'|_g^3 \\ -2i|\omega'|_g - ia'(x,\omega') & -i|\omega'|_g^2 \end{pmatrix}
$$
$$
= -i|\omega'|_g^3 \big(2a'(x,\omega') + 3|\omega'|_g\big) \neq 0,
$$

if $\omega' \neq 0$ since $2a'(x,\omega') + 3|\omega'|_g \neq 0$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 3.2.

4. Lopatinskiĭ-Šapiro condition for the conjugated bi-Laplacian

Set $P_\sigma = \Delta^2 - \sigma^4$ with $\sigma \in [0, +\infty)$ and denote by $P_{\sigma,\varphi} = e^{\tau\varphi} P_\sigma e^{-\tau\varphi}$ the conjugate operator of P_σ with $\tau \geqslant 0$ a large parameter and $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d, \mathbb{R})$. We shall refer to φ as the weight function. The principal symbol of P_σ in normal geodesic coordinates is given by

$$p_\sigma(x,\xi) = (\xi_d^2 + r(x,\xi'))^2 - \sigma^4.$$

Observe that $e^{\tau\varphi}D_j e^{-\tau\varphi} = D_j + i\tau\partial_j\varphi \in \mathscr{D}_\tau^1$. So, after conjugation, the principal symbol becomes

$$
\begin{aligned}
p_{\sigma,\varphi}(x,\xi,\tau) &= p_\sigma(x, \xi + i\tau d_x\varphi) \\
&= \left((\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi)\right)^2 - \sigma^4 \\
&= \left((\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi) - \sigma^2\right) \\
&\qquad \times \left((\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi) + \sigma^2\right).
\end{aligned}
$$

We write $p_{\sigma,\varphi}(x,\xi,\tau) = q_{\sigma,\varphi}^1(x,\xi,\tau)q_{\sigma,\varphi}^2(x,\xi,\tau)$ with

$$
q_{\sigma,\varphi}^j(x,\xi,\tau) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2, \quad j = 1,2.
$$

We consider two boundary operators B_1 and B_2 of order k_1 and k_2 with $b_j(x,\xi)$ for principal symbol, $j = 1,2$. The associated conjugated operators

$$
B_{j,\varphi} = e^{\tau\varphi}B_j e^{-\tau\varphi},
$$

have respective principal symbols

$$
b_{j,\varphi}(x,\xi,\tau) = b_j(x, \xi + i\tau d\varphi), \quad j = 1,2.
$$

We assume that the Lopatinskiĭ-Šapiro condition holds for (P_0, B_1, B_2) as in Definition 3.1 for any point $(x,\omega') \in T_x^*\partial\Omega$. We wish to know if the Lopatinskiĭ-Šapiro condition can hold for $(P_\sigma, B_1, B_2, \varphi)$, as given by the following definition (in the chosen local normal geodesic coordinates for simplicity).

Definition 4.1. — Let $(x,\xi',\tau,\sigma) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0,+\infty) \times [0,+\infty)$ with $(\xi',\tau,\sigma) \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at (x,ξ',τ,σ) if for any polynomial $f(z)$ with complex coefficients there exist $c_1, c_2 \in \mathbb{C}$ and a polynomial $\ell(z)$ with complex coefficients such that, for all $z \in \mathbb{C}$

$$
f(z) = c_1 b_{1,\varphi}(x,\xi',\xi_d = z,\tau) + c_2 b_{2,\varphi}(x,\xi',\xi_d = z,\tau) + \ell(z)p_{\sigma,\varphi}^+(x,\xi',\xi_d = z,\tau),
$$

with

$$
p_{\sigma,\varphi}^+(x,\xi',\xi_d = z,\tau) = \prod_{\operatorname{Im}\rho_j(\xi',\tau,\sigma)\geqslant 0} (z - \rho_j(\xi',\tau,\sigma)),
$$

where $\rho_j(x,\xi',\tau,\sigma)$, $j = 1,\ldots,4$, are the complex roots of the polynomial $z \mapsto p_{\sigma,\varphi}(x,\xi',\xi_d = z,\tau)$.

In what follows, we shall assume that $\partial_d\varphi > 0$. Locally, one has $\partial_d\varphi \geqslant C_1 > 0$, for some $C_1 > 0$.

4.1. Lopatinskiĭ-Šapiro condition and root positions. — With the assumption that $\partial_d\varphi > 0$, for any point (x,ξ',τ,σ) at most two roots lie in the upper complex closed half-plane (this is explained below). We then enumerate the following cases.

  – Case 1 : No root lying in the upper complex closed half-plane, then

$$
p_{\sigma,\varphi}^+(x,\xi',\xi_d,\tau) = 1
$$

and the Lopatinskiĭ-Šapiro condition of Definition 4.1 holds trivially at (x,ξ',τ,σ).

– Case 2 : One root lying in the upper complex closed half-plane. Let us denote by ρ^+ that root, then $p^+_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = \xi_d - \rho^+$. With Definition 4.1, for any choice of f, the polynomial $\xi_d \mapsto f(\xi_d) - c_1 b_{1,\varphi}(x,\xi',\xi_d,\tau) - c_2 b_{2,\varphi}(x,\xi',\xi_d,\tau)$ admits ρ^+ as a root for $c_1, c_2 \in \mathbb{C}$ well chosen. Hence, the Lopatinskiĭ-Šapiro condition holds at (x,ξ',τ,σ) if and only if

$$b_{1,\varphi}(x,\xi',\xi_d = \rho^+,\tau) \neq 0 \ \text{ or } \ b_{2,\varphi}(x,\xi',\xi_d = \rho^+,\tau) \neq 0.$$

Note that it then suffices to have

$$\det \begin{pmatrix} b_{1,\varphi} & b_{2,\varphi} \\ \partial_{\xi_d} b_{1,\varphi} & \partial_{\xi_d} b_{2,\varphi} \end{pmatrix}(x,\xi',\xi_d = \rho^+,\tau) \neq 0.$$

– Case 3 : Two different roots lying in the upper complex closed half-plane. Let denote by ρ^+_1 and ρ^+_2 these roots. One has $p^+_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = (\xi_d - \rho^+_1)(\xi_d - \rho^+_2)$. The Lopatinskiĭ-Šapiro condition holds at (x,ξ',τ,σ) if and only if the complex numbers ρ^+_1 and ρ^+_2 are the roots of the polynomial

$$\xi_d \longmapsto f(\xi_d) - c_1 b_{1,\varphi}(x,\xi',\xi_d,\tau) - c_2 b_{2,\varphi}(x,\xi',\xi_d,\tau),$$

for c_1, c_2 well chosen. This reads

$$\begin{cases} f(\rho^+_1) = c_1 b_{1,\varphi}(x,\xi',\xi_d = \rho^+_1,\tau) + c_2 b_{2,\varphi}(x,\xi',\xi_d = \rho^+_1,\tau), \\ f(\rho^+_2) = c_1 b_{1,\varphi}(x,\xi',\xi_d = \rho^+_2,\tau) + c_2 b_{2,\varphi}(x,\xi',\xi_d = \rho^+_2,\tau). \end{cases}$$

Being able to solve this system in c_1 and c_2 for any f is equivalent to having

$$(4.1) \qquad \det \begin{pmatrix} b_{1,\varphi}(x,\xi',\xi_d = \rho^+_1,\tau) \ b_{2,\varphi}(x,\xi',\xi_d = \rho^+_1,\tau) \\ b_{1,\varphi}(x,\xi',\xi_d = \rho^+_2,\tau) \ b_{2,\varphi}(x,\xi',\xi_d = \rho^+_2,\tau) \end{pmatrix} \neq 0.$$

– Case 4 : A double root lying in the upper complex closed half-plane. Denote by ρ^+ this root; one has $p^+_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = (\xi_d - \rho^+)^2$. The Lopatinskiĭ-Šapiro condition holds at at (x,ξ',τ,σ) if and only if for any choice of f, the complex number ρ^+ is a double root of the polynomial $\xi_d \mapsto f(\xi_d) - c_1 b_{1,\varphi}(x,\xi',\xi,\tau) - c_2 b_{2,\varphi}(x,\xi',\xi,\tau)$ for some $c_1, c_2 \in \mathbb{C}$. Thus having the Lopatinskiĭ-Šapiro condition is equivalent of having the following determinant condition,

$$(4.2) \qquad \det \begin{pmatrix} b_{1,\varphi}(x,\xi',\xi_d = \rho^+,\tau) & b_{2,\varphi}(x,\xi',\xi_d = \rho^+,\tau) \\ \partial_{\xi_d} b_{1,\varphi}(x,\xi',\xi_d = \rho^+,\tau) \ \partial_{\xi_d} b_{2,\varphi}(x,\xi',\xi_d = \rho^+,\tau) \end{pmatrix} \neq 0.$$

Observe that Case 4 can only occur if $\sigma = 0$ (then one has $(\xi',\tau) \neq (0,0)$). If $\sigma > 0$ then only Cases 1, 2, and 3 are possible. This is precisely stated in Lemma 4.7. This will be an important point in what follows.

We now state the following important proposition.

Proposition 4.2. — *Let $x^0 \in \partial\Omega$. Assume that the Lopatinskiĭ-Šapiro condition holds for (P_0, B_1, B_2) at x^0 and thus in a compact neighborhood V^0 of x^0 (by Remark 3.3). Assume also that $\partial_d \varphi \geqslant C_1 > 0$ in V^0. There exist $\mu_0 > 0$ and $\mu_1 > 0$ such that if $(x,\xi',\tau,\sigma) \in V^0 \times \mathbb{R}^{d-1} \times [0,+\infty) \times [0,+\infty)$ with $(\xi',\tau,\sigma) \neq (0,0,0)$,*

$$|d_{x'}\varphi(x)| \leqslant \mu_0 \partial_d \varphi(x) \ \text{ and } \ \sigma \leqslant \mu_1 \tau \partial_d \varphi(x),$$

then the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at (x,ξ',τ,σ).

The proof of Proposition 4.2 is given below. We first need to analyze the configuration of the roots of the symbol $p_{\sigma,\varphi}$ starting with each factor $q_{\sigma,\varphi}^j$, $j = 1, 2$.

4.2. ROOT CONFIGURATION FOR EACH FACTOR. — We consider either factors $\xi_d \mapsto q_{\sigma,\varphi}^j(x, \xi', \xi_d, \tau)$. We recall that

$$q_{\sigma,\varphi}^j(x, \xi, \tau) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2, \quad j = 1, 2.$$

First, we consider the case $r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \in \mathbb{R}^-$, that is, equal to $-\beta^2$ with $\beta \in \mathbb{R}$. Then, the roots of $\xi_d \mapsto q_{\sigma,\varphi}^j(x, \xi', \xi_d, \tau)$ are given by

$$-i\tau\partial_d\varphi + \beta \quad \text{and} \quad -i\tau\partial_d\varphi - \beta.$$

Both lie in the lower complex open half-plane.

Second, we consider the case $r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \in \mathbb{C} \setminus \mathbb{R}^-$. There exists a unique $\alpha_j \in \mathbb{C}$ such that $\operatorname{Re}\alpha_j > 0$ and

$$
\begin{aligned}
(4.3) \quad \alpha_j^2(x, \xi', \tau, \sigma) &= r(x, \xi' + i\tau d_{x'}\varphi(x)) + (-1)^j\sigma^2 \\
&= r(x, \xi') - \tau^2 r(x, d_{x'}\varphi(x)) + (-1)^j\sigma^2 + i2\tau\widetilde{r}(x, \xi', d_{x'}\varphi(x))^2,
\end{aligned}
$$

where $\widetilde{r}(x, ., .)$ denotes the symmetric bilinear form associated with the quadratic form $r(x, .)$. Then, the two roots of $\xi_d \mapsto q_{\sigma,\varphi}^j(x, \xi', \xi_d, \tau)$ are given by

$$(4.4) \qquad \pi_{j,1}(x, \xi', \tau, \sigma) = -i\tau\partial_d\varphi(x) - i\alpha_j(x, \xi', \tau, \sigma)$$

and

$$(4.5) \qquad \pi_{j,2}(x, \xi', \tau, \sigma) = -i\tau\partial_d\varphi(x) + i\alpha_j(x, \xi', \tau, \sigma).$$

One has $\operatorname{Im}\pi_{j,1} < 0$ since $\partial_d\varphi \geqslant C_1 > 0$. With $\operatorname{Im}\pi_{j,2} = -\tau\partial_d\varphi + \operatorname{Re}\alpha_j$ one sees that the sign of $\operatorname{Im}\pi_{j,2}$ may change. The following lemma gives an algebraic characterization of the sign of $\operatorname{Im}\pi_{j,2}$.

LEMMA 4.3. — *Assume that $\partial_d\varphi > 0$. Having $\operatorname{Im}\pi_{j,2}(x, \xi', \tau, \sigma) < 0$ is equivalent to having*

$$(\partial_d\varphi)^2 r(x, \xi') + \widetilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2 |d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2.$$

Proof. — From (4.4)–(4.5) one has $\operatorname{Im}\pi_{j,2} < 0$ if and only if $\operatorname{Re}\alpha_j < \tau\partial_d\varphi = |\tau\partial_d\varphi|$, that is, if and only if

$$4(\tau\partial_d\varphi)^2 \operatorname{Re}\alpha_j^2 - 4(\tau\partial_d\varphi)^4 + (\operatorname{Im}\alpha_j^2)^2 < 0,$$

by Lemma 4.4 below. With (4.3) this gives the result.   □

LEMMA 4.4. — *Let $z \in \mathbb{C}$ such that $m = z^2$. For $x_0 \in \mathbb{R}$ such that $x_0 \neq 0$, we have*

$$|\operatorname{Re}z| \lessgtr_\eqq |x_0| \iff 4x_0^2 \operatorname{Re}m - 4x_0^4 + (\operatorname{Im}m)^2 \lessgtr_\eqq 0.$$

The notation $\lessgtr_\eqq$ is to be understood as $<$, $=$, or $>$ in each term of the equivalence respectively.

Proof. — Let $z = x + iy \in \mathbb{C}$. On the one hand we have $z^2 = x^2 - y^2 + 2ixy = m$ and $\operatorname{Re} m = x^2 - y^2$, $\operatorname{Im} m = 2xy$. On the other hand we have

$$4x_0^2 \operatorname{Re} m - 4x_0^4 + (\operatorname{Im} m)^2 = 4x_0^2(x^2 - y^2) - 4x_0^4 + 4x^2 y^2$$
$$= 4(x_0^2 + y^2)(x^2 - x_0^2),$$

thus with the same sign as $(x^2 - x_0^2)$. Since $|\operatorname{Re} z| \gtreqqless |x_0| \Leftrightarrow x^2 - x_0^2 \gtreqqless 0$, the conclusion follows. $\qquad\square$

With the following two lemmas we now connect the sign of $\operatorname{Im} \pi_{j,2}$ with the low frequency regime $|\xi'| \lesssim \tau$.

Lemma 4.5. — *Assume there exists $K_0 > 0$ such that $|d_{x'}\varphi| \leqslant K_0|\partial_d\varphi|$. Then, there exists $C_{K_0} > 0$ such that $\operatorname{Im} \pi_{j,2}(x, \xi', \tau, \sigma) < 0$ if $C_{K_0}|\xi'| + \sigma \leqslant \tau\partial_d\varphi$, $j = 0, 1$.*

Proof. — With Lemma 4.3 having $\operatorname{Im} \pi_{j,2} < 0$ reads

$$(4.6) \qquad (\partial_d\varphi)^2 r(x, \xi') + \widetilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2.$$

On the one hand, since $|d_{x'}\varphi| \leqslant K_0|\partial_d\varphi|$ one has

$$(\partial_d\varphi)^2 r(x, \xi') + \widetilde{r}(x, \xi', d_{x'}\varphi)^2 \leqslant K(\partial_d\varphi)^2|\xi'|^2,$$

for some $K > 0$ that depends on K_0, using that $|\xi'|_x \eqsim |\xi'|$. On the other hand one has

$$\tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2 \geqslant \tau^2(\partial_d\varphi)^4 - \sigma^2(\partial_d\varphi)^2.$$

Thus (4.6) holds if one has

$$\tau^2(\partial_d\varphi)^4 - \sigma^2(\partial_d\varphi)^2 \geqslant K(\partial_d\varphi)^2|\xi'|^2,$$

that is, $\tau^2(\partial_d\varphi)^2 \geqslant K|\xi'|^2 + \sigma^2$. $\qquad\square$

Lemma 4.6. — *Let W be a bounded open set of $\mathbb{R}^d$ and $x^0 \in W$. Assume that $\partial_d\varphi > 0$ in $\overline{W}$ and let $\kappa_0 > 0$. Then, there exists $C > 0$ such that*

$$|\xi'| \leqslant C\tau \quad \textit{if } \operatorname{Im} \pi_{j,2}(x, \xi', \tau, \sigma) < 0 \textit{ and } \kappa_0\sigma \leqslant \tau, \quad x \in W.$$

Proof. — With Lemma 4.3 having $\operatorname{Im} \pi_{j,2} < 0$ reads

$$(\partial_d\varphi)^2 r(x, \xi') + \widetilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2.$$

In particular, this implies

$$r(x, \xi') < \tau^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2 \leqslant (\sup_W |d_x\varphi|_x^2 + 1/\kappa_0^2)\tau^2.$$

The result follows since $|\xi'| \asymp r(x, \xi')$. $\qquad\square$

As mentioned in Section 4.1, we have the following result.

Lemma 4.7. — *Assume that $\sigma > 0$. Then, $\pi_{1,2}(x, \xi', \tau, \sigma) \neq \pi_{2,2}(x, \xi', \tau, \sigma)$. Moreover, the roots $\pi_{1,2}(x, \xi', \tau, \sigma)$ and $\pi_{2,2}(x, \xi', \tau, \sigma)$ cannot be both real.*

Proof. — With the forms of the roots given in (4.4)–(4.5) if $\pi_{1,2} = \pi_{2,2}$ then $\alpha_1 = \alpha_2$, thus $\alpha_1^2 = \alpha_2^2$ implying $\sigma^2 = 0$.

Assume now that $\pi_{1,2} \in \mathbb{R}$ and $\pi_{2,2} \in \mathbb{R}$, that is, $\operatorname{Im} \pi_{1,2} = \operatorname{Im} \pi_{2,2} = 0$. This reads $\operatorname{Re} \alpha_j = \tau \partial_d \varphi$, giving $|\operatorname{Re} \alpha_j| = |\partial_d \varphi|$, for $j = 1$ and 2. With Lemma 4.4 one has

$$4(\tau \partial_d \varphi)^2 \operatorname{Re} \alpha_j^2 - 4(\tau \partial_d \varphi)^4 + (\operatorname{Im} \alpha_j^2)^2 = 0, \qquad j = 1, 2.$$

Observing that $\operatorname{Im} \alpha_1^2 = \operatorname{Im} \alpha_2^2$ one thus obtains $\operatorname{Re} \alpha_1^2 = \operatorname{Re} \alpha_2^2$, and the conclusion follows as for the first part. □

4.3. **Proof of Proposition 4.2.** — Here, according to the statement of Proposition 4.2 we consider

$$|d_{x'}\varphi| \leqslant \mu_0 \partial_d \varphi \quad \text{and} \quad \sigma \leqslant \mu_1 \tau \partial_d \varphi.$$

First, we choose $0 < \mu_0 \leqslant 1$ and $0 < \mu_1 \leqslant 1/2$. Below both may be chosen much smaller. According to Lemma 4.5, with $K_0 = 1$ therein, for some $C_2 = 2C_{K_0} > 0$ if one has $C_2 |\xi'| \leqslant \tau \partial_d \varphi$ then all four roots of $\xi_d \mapsto p_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ lie in the lower complex open half-plane. If so, we face Case 1 as in the discussion of Section 4.1 and the Lopatinskiĭ-Šapiro condition holds. To carry on with the proof of Proposition 4.2 we now only have to consider having

$$(4.7) \qquad\qquad \tau \partial_d \varphi \leqslant C_2 |\xi'|.$$

Our proof of Proposition 4.2 relies on the following lemma.

Lemma 4.8. — *There exists $C_3 > 0$ such that, for $j = 1$ or 2, for $0 < \mu_0 \leqslant 1$, $0 < \mu_1 \leqslant 1/2$, and for all $(x, \xi', \tau, \sigma) \in \overline{V^0} \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$, one has*

$$|d_{x'}\varphi(x)| \leqslant \mu_0 \partial_d \varphi(x), \quad \sigma \leqslant \mu_1 \tau \partial_d \varphi(x) \quad \text{and} \quad \operatorname{Im} \pi_{j,2}(x, \xi', \tau, \sigma) \geqslant 0$$
$$\implies |\alpha_j - |\xi'|_x| + \tau |d_{x'}\varphi(x)| \leqslant |\xi'|_x C_3(\mu_0 + \mu_1^2).$$

Proof. — With (4.7) one has

$$(4.8) \qquad\qquad \tau |d_{x'}\varphi| \leqslant \mu_0 \tau \partial_d \varphi \lesssim \mu_0 |\xi'|_x.$$

With the first-order Taylor formula one has

$$\alpha_j^2 = r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j \sigma^2$$
$$= r(x, \xi') + \int_0^1 d_{\xi'} r(x, \xi' + i\tau s \, d_{x'}\varphi)(i\tau d_{x'}\varphi) ds + (-1)^j \sigma^2.$$

With (4.8) and homogeneity one has

$$\left| d_{\xi'} r(x, \xi' + i\tau s \, d_{x'}\varphi)(i\tau d_{x'}\varphi) \right| \lesssim \mu_0 |\xi'|_x^2.$$

One also has $\sigma \leqslant \mu_1 \tau \partial_d \varphi \lesssim \mu_1 |\xi'|_x$. Since $r(x, \xi') = |\xi'|_x^2$, this yields

$$\alpha_j^2 = |\xi'|_x^2 \left(1 + O(\mu_0 + \mu_1^2)\right) \quad \text{and hence} \quad \alpha_j = |\xi'|_x \left(1 + O(\mu_0 + \mu_1^2)\right).$$

This and (4.8) give the result. □

Before proceeding, we make the following computation. For $j = 1, 2$ and $\ell = 1, 2$ we write

$$
\begin{aligned}
(4.9) \qquad b_{\ell,\varphi}(x, \xi', \xi_d = \pi_{j,2}, \tau) &= b_\ell(x, \xi' + i\tau d_{x'}\varphi, \pi_{j,2} + i\tau\partial_d\varphi) \\
&= b_\ell(x, \xi' + i\tau d_{x'}\varphi, i\alpha_j) \\
&= b_\ell(x, \xi' + i\tau d_{x'}\varphi, i|\xi'|_x + i(\alpha_j - |\xi'|_x)).
\end{aligned}
$$

We use Lemma 3.4 and the value of $\varepsilon > 0$ given therein. We choose $0 < \mu_0 \leqslant 1$ and $0 < \mu_1 \leqslant 1/2$ such that

$$
(4.10) \qquad\qquad\qquad C_3(\mu_0 + \mu_1^2) \leqslant \varepsilon,
$$

with $C_3 > 0$ as given by Lemma 4.8.

We now consider the root configurations that remain to consider according to the discussion in Section 4.1.

Case 2. — In this case, one root of $p_{\sigma,\varphi}$ lies in the upper complex closed half-plane. We denote this root by ρ^+. According to the discussion in Section 4.1 it suffices to prove that

$$
(4.11) \qquad\qquad \det \begin{pmatrix} b_{1,\varphi} & b_{2,\varphi} \\ \partial_{\xi_d} b_{1,\varphi} & \partial_{\xi_d} b_{2,\varphi} \end{pmatrix} (x, \xi', \xi_d = \rho^+, \tau) \neq 0.
$$

In fact, one has $\rho^+ = \pi_{j,2}$ with $j = 1$ or 2. We use the first part of Lemma 3.4 with $\zeta' = i\tau d_{x'}\varphi$ and $\delta = i(\alpha_j - |\xi|_x)$. With (4.9) and (4.10) with Lemma 4.8 and the first part of Lemma 3.4 one obtains (4.11).

Case 3. — In this case $\operatorname{Im}\pi_{1,2} > 0$ and $\operatorname{Im}\pi_{2,2} > 0$. According to the discussion in Section 4.1 it suffices to prove that

$$
(4.12) \qquad\quad \det \begin{pmatrix} b_{1,\varphi}(x, \xi', \xi_d = \pi_{1,2}, \tau) \; b_{2,\varphi}(x, \xi', \xi_d = \pi_{1,2}, \tau) \\ b_{1,\varphi}(x, \xi', \xi_d = \pi_{2,2}, \tau) \; b_{2,\varphi}(x, \xi', \xi_d = \pi_{2,2}, \tau) \end{pmatrix} \neq 0.
$$

We use the second part of Lemma 3.4 with $\zeta' = i\tau d_{x'}\varphi$, $\delta = i(\alpha_1 - |\xi|_x)$, and $\widetilde{\delta} = i(\alpha_2 - |\xi|_x)$. With (4.9) and (4.10) with Lemma 4.8 and the second part of Lemma 3.4 one obtains (4.12).

Case 4. — In this case (that only occurs if $\sigma = 0$) the Lopatinskiĭ-Šapiro condition holds also if one has (4.11). The proof is thus the same as for Case 2. This concludes the proof of Proposition 4.2. $\qquad\qquad\square$

4.4. Local stability of the algebraic conditions. — In Section 3 we saw that the Lopatinskiĭ-Šapiro condition for (P_σ, B_1, B_2) in Definition 3.1 exhibits some stability property. This was used in the statement of Proposition 4.2 that states how the Lopatinskiĭ-Šapiro condition for (P_σ, B_1, B_2) can imply the Lopatinskiĭ-Šapiro condition of Definition 4.1 for $(P_\sigma, B_1, B_2, \varphi)$, that is, the version of this condition for the conjugated operators.

A natural question would then be: does the Lopatinskiĭ-Šapiro condition for the conjugated operators enjoy the same stability property? The answer is yes. Yet, this

is not needed in what follows. In fact, below one exploits the algebraic conditions listed in Section 4.1 once the Lopatinskiĭ-Šapiro condition is known to hold at a point $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ in tangential phase space. One thus rather needs to know that these algebraic conditions are stable. Here also the answer is positive and is the subject of the present section.

As in Definition 4.1 for $\varrho' = (x, \xi', \tau, \sigma)$ one denotes by $\rho_j(\varrho')$ the roots of $p_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ viewed as a polynomial in ξ_d.

Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$. One sets

$$J^+ = \left\{ j \in \{1, 2, 3, 4\} \mid \operatorname{Im} \rho_j(\varrho^{0\prime}) \geqslant 0 \right\},$$
$$J^- = \left\{ j \in \{1, 2, 3, 4\} \mid \operatorname{Im} \rho_j(\varrho^{0\prime}) < 0 \right\},$$

and, for $\varrho' = (x, \xi', \tau, \sigma)$,

$$\kappa_{\varrho^{0\prime}}^+(\varrho') = \prod_{j \in J^+} \left(\xi_d - \rho_j(\varrho') \right), \qquad \kappa_{\varrho^{0\prime}}^-(\varrho') = \prod_{j \in J^-} \left(\xi_d - \rho_j(\varrho') \right).$$

Naturally, one has

$$\kappa_{\varrho^{0\prime}}^+(\varrho^{0\prime}, \xi_d) = p_{\sigma,\varphi}^+(x^0, \xi^{0\prime}, \xi_d, \tau^0) \quad \text{and} \quad \kappa_{\varrho^{0\prime}}^-(\varrho^{0\prime}, \xi_d) = p_{\sigma,\varphi}^-(x^0, \xi^{0\prime}, \xi_d, \tau^0).$$

Moreover, there exists a conic open neighborhood $\mathcal{U}_0$ of $\varrho^{0\prime}$ where both $\kappa_{\varrho^{0\prime}}^+(\varrho')$ and $\kappa_{\varrho^{0\prime}}^-(\varrho')$ are smooth with respect to ϱ'. One has

$$p_{\sigma,\varphi} = p_{\sigma,\varphi}^+ p_{\sigma,\varphi}^- = \kappa_{\varrho^{0\prime}}^+ \kappa_{\varrho^{0\prime}}^-.$$

Note however that for $\varrho' = (x, \xi', \tau, \sigma) \in \mathcal{U}_0$ it may very well happen that

$$p_{\sigma,\varphi}^+(x, \xi', \xi_d, \tau) \neq \kappa_{\varrho^{0\prime}}^+(\varrho', \xi_d) \quad \text{and} \quad p_{\sigma,\varphi}^-(x, \xi', \xi_d, \tau) \neq \kappa_{\varrho^{0\prime}}^-(\varrho', \xi_d).$$

The following proposition can be found in [8, Prop. 1.8].

PROPOSITION 4.9. — *Let the Lopatinskiĭ-Šapiro condition hold at*

$$\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$$

for $(P_\sigma, B_1, B_2, \varphi)$*. Then,*

(1) *The polynomial* $\xi_d \mapsto p_{\sigma,\varphi}^+(x^0, \xi^{0\prime}, \xi_d, \tau^0)$ *is of degree less than or equal to two.*

(2) *There exists a conic open neighborhood* $\mathcal{U}$ *of* $\varrho^{0\prime}$ *such that the family* $\{b_\varphi^1(\varrho', \xi_d), b_\varphi^2(\varrho', \xi_d)\}$ *is complete modulo* $\kappa_{\varrho^{0\prime}}^+(\varrho', \xi_d)$ *at every point* $\varrho' = (x, \xi', \tau, \sigma) \in \mathcal{U}$, *namely for any polynomial* $f(\xi_d)$ *with complex coefficients there exist* $c_1, c_2 \in \mathbb{C}$ *and a polynomial* $\ell(\xi_d)$ *with complex coefficients such that, for all* $\xi_d \in \mathbb{C}$

$$(4.13) \qquad f(\xi_d) = c_1 b_{1,\varphi}(x, \xi', \xi_d, \tau) + c_2 b_{2,\varphi}(x, \xi', \xi_d, \tau) + \ell(\xi_d) \kappa_{\varrho^{0\prime}}^+(\varrho', \xi_d).$$

We emphasize again that the second property in Proposition 4.9 looks very much like the statement of Lopatinskiĭ-Šapiro condition for $(P_\sigma, B_1, B_2, \varphi)$ at ϱ' in Definition 4.1. Yet, it differs by having $p_{\sigma,\varphi}^+(x, \xi', \xi_d, \tau)$ that only depends on the root configuration at ϱ' replaced by $\kappa_{\varrho^{0\prime}}^+(\varrho', \xi_d)$ whose structure is based on the root configuration at $\varrho^{0\prime}$.

Let m^+ be the common degree of $p^+_{\sigma,\varphi}(\varrho^{0\prime}, \xi_d)$ and $\kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d)$ and m^- be the common degree of $p^-_{\sigma,\varphi}(\varrho^{0\prime}, \xi_d)$ and $\kappa^-_{\varrho^{0\prime}}(\varrho', \xi_d)$ for $\varrho' \in \mathscr{U}$. One has $m^+ + m^- = 4$ and thus $m^- \geqslant 2$ for $\varrho' \in \mathscr{U}$ since $m^+ \leqslant 2$.

Invoking the Euclidean division of polynomials, one sees that it is sufficient to consider polynomials f of degree less than or equal to $m^+ - 1 \leqslant 1$ in (4.13). Since the degree of $b_{j,\varphi}(\varrho', \xi_d)$ can be as high as $3 > m^+ - 1$ it however makes sense to consider f of degree less than or equal to $m = 3$. Then, the second property in Proposition 4.9 is equivalent to having

$$\{b_{1,\varphi}(x, \xi', \xi_d, \tau), b_{2,\varphi}(x, \xi', \xi_d, \tau)\} \cup \bigcup_{0 \leqslant \ell \leqslant 3 - m^+} \{\kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d)\xi_d^\ell\}$$

be a complete in the set of polynomials of degree less than or equal to $m = 3$. Note that this family is made of $m' = 6 - m^+ = 2 + m^-$ polynomials.

We now express an inequality that follows from Proposition 4.9 that will be key in the boundary estimation given in Proposition 5.1 below.

4.5. Symbol positivity at the boundary. — The symbols $b_{j,\varphi}$, $j = 1, 2$, are polynomial in ξ_d of degree $k_j \leqslant 3$ and we may thus write them in the form

$$b_{j,\varphi}(\varrho', \xi_d) = \sum_{\ell=0}^{k_j} b^\ell_{j,\varphi}(\varrho')\xi_d^\ell,$$

with $b^\ell_{j,\varphi}$ homogeneous of degree $k_j - \ell$.

The polynomial $\xi_d \mapsto \kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d)$ is of degree $m^+ \leqslant 2$ for $\varrho' \in \mathscr{U}$ with $\mathscr{U}$ given by Proposition 4.9. Similarly, we write

$$\kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d) = \sum_{\ell=0}^{m^+} \kappa^{+,\ell}_{\varrho^{0\prime}}(\varrho')\xi_d^\ell,$$

with $\kappa^{+,\ell}_{\varrho^{0\prime}}$ homogeneous of degree $m^+ - \ell$. We introduce

$$e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \begin{cases} b_{j,\varphi}(\varrho', \xi_d) & \text{if } j = 1, 2, \\ \kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d)\xi_d^{j-3} & \text{if } j = 3, \ldots, m'. \end{cases}$$

As explained above, all these polynomials are of degree less than or equal to three. If we now write

$$e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \sum_{\ell=0}^{3} e^\ell_{j,\varrho^{0\prime}}(\varrho')\xi_d^\ell,$$

for $j = 1, 2$ one has $e^\ell_{j,\varrho^{0\prime}}(\varrho') = b^\ell_{j,\varphi}(\varrho')$, with $\ell = 0, \ldots, k_j$ and $e^\ell_{j,\varrho^{0\prime}}(\varrho') = 0$ for $\ell > k_j$, and for $j = 3, \ldots, m'$,

$$e^\ell_{j,\varrho^{0\prime}}(\varrho') = \begin{cases} 0 & \text{if } \ell < j - 3, \\ \kappa^{+,\ell+3-j}_{\varrho^{0\prime}}(\varrho') & \text{if } \ell = j - 3, \ldots, m^+ + j - 3 \leqslant m^+ + m' - 3, \\ 0 & \text{if } \ell > m^+ + j - 3. \end{cases}$$

In particular $e^\ell_{j,\varrho^{0\prime}}(\varrho')$ is homogeneous of degree $m^+ + j - \ell - 3$. Note that $m^+ + m' - 3 = 3$. We thus have the following result.

Lemma 4.10. — *Set the $m' \times (m+1)$ matrix $M(\varrho') = (M_{j,\ell}(\varrho'))_{\substack{1 \leqslant j \leqslant m' \\ 0 \leqslant \ell \leqslant m}}$ with $M_{j,\ell}(\varrho') = e^\ell_{j,\varrho^{0\prime}}(\varrho')$. Then, the second point in Proposition 4.9 states that $M(\varrho')$ is of rank $m + 1 = 4$ for $\varrho' \in \mathscr{U}$.*

Recall that $m' = m^- + 2 \geqslant 4$. We now set

$$(4.14) \qquad \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) = \sum_{\ell=0}^3 e^\ell_{j,\varrho^{0\prime}}(\varrho') z_\ell = \sum_{\ell=0}^3 M_{j,\ell}(\varrho') z_\ell, \qquad \boldsymbol{z} = (z_0, \dots, z_3).$$

in agreement with the notation introduced in (2.3) in Section 2.4.2. One has the following positivity result.

Lemma 4.11. — *Let the Lopatinskiĭ-Šapiro condition hold at a point*

$$\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$$

for $(P_\sigma, B_1, B_2, \varphi)$ and let $\mathscr{U}$ be as given by Proposition 4.9. Then, if $\varrho' \in \mathscr{U}$ there exists $C > 0$ such that

$$\sum_{j=1}^{m'} \big| \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) \big|^2 \geqslant C |\boldsymbol{z}|^2_{\mathbb{C}^4}, \qquad \boldsymbol{z} = (z_0, \dots, z_3) \in \mathbb{C}^4.$$

Proof. — In $\mathbb{C}^4$ define the bilinear form

$$\Sigma_{\mathscr{B}}(\boldsymbol{z}, \boldsymbol{z}') = \sum_{j=1}^{m'} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) \overline{\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}')}.$$

With (4.14) one has

$$\Sigma_{\mathscr{B}}(\boldsymbol{z}, \boldsymbol{z}') = \big(M(\varrho')\boldsymbol{z}, M(\varrho')\boldsymbol{z}' \big)_{\mathbb{C}^{m'}} = \big({}^t\overline{M(\varrho')} M(\varrho')\boldsymbol{z}, \boldsymbol{z}' \big)_{\mathbb{C}^4}.$$

As rank ${}^t\overline{M(\varrho')} M(\varrho') = \operatorname{rank} M(\varrho') = 4$ by Lemma 4.10 one obtains the result. $\qquad\square$

5. Boundary norm estimate under Lopatinskiĭ-Šapiro condition

Near $x^0 \in \partial\Omega$ we consider two boundary operators B_1 and B_2. As in Section 4 the associated conjugated operators are denoted by $B_{j,\varphi}$, $j = 1, 2$ with respective principal symbols $b_{j,\varphi}(x, \xi, \tau)$.

The main result of this section is the following proposition for the fourth-order conjugated operator $P_{\sigma,\varphi}$. It is key in the final result of the present article. It states that all traces are controlled by norms of $B_{1,\varphi} v_{|x_d=0^+}$ and $B_{2,\varphi} v_{|x_d=0^+}$ if the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2, φ).

Proposition 5.1. — *Let $\kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$. Assume that $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geqslant \kappa_0 \sigma$.*

Then, there exist a neighborhood W^0 of x^0, $C > 0$, $\tau_0 > 0$ such that

$$|\operatorname{tr}(v)|_{3,1/2,\tau} \leqslant C\big(\|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau}\big),$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_0, \kappa_0\sigma)$ and $v \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The notation of the function space $\overline{\mathscr{C}}_c^\infty(W_+^0)$ is introduced in (1.6). For the proof of Proposition 5.1 we start with a microlocal version of the result.

5.1. A microlocal estimate

PROPOSITION 5.2. — *Let $\kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let*

$$(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$$

nonvanishing with $\tau^0 \geqslant \kappa_1 \sigma^0$ and such that $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.1 at $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$.

Then, there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0\sigma$ such that if $\chi \in S^0_{\tau,\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$|\operatorname{tr}(\operatorname{Op}_\tau(\chi)v)|_{3,1/2,\tau} \leqslant C\bigg(\sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi}v\|_+$$
$$+ \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau}\bigg),$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_0, \kappa_0\sigma)$ and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

Proof. — We choose a conic open neighborhood $\mathscr{U}_0$ of $\varrho^{0\prime}$ according to Proposition 4.9 and such that $\mathscr{U}_0 \subset W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$. Assume moreover that $\tau \geqslant \kappa_0\sigma$ in $\mathscr{U}_0$.

In Section 4.5 we introduced the symbols $e_{j,\varrho^{0\prime}}(\varrho', \xi_d)$, $j = 1, \ldots, m' = m^- + 2 = 6 - m^+$. For a conic set $\mathscr{V}$ denote $\mathbb{S}_{\mathscr{V}} = \{\varrho' = (x, \xi', \tau, \sigma) \in \mathscr{V} \mid |(\xi', \tau, \sigma)| = 1\}$.

Consequence of the Lopatinskiĭ-Šapiro condition holding at $\varrho^{0\prime}$, by Lemma 4.11 for all $\varrho' \in \mathbb{S}_{\mathscr{U}_0}$ there exists $C > 0$ such that

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geqslant C|\mathbf{z}|_{\mathbb{C}^4}^2, \qquad \mathbf{z} = (z_0, \ldots, z_3) \in \mathbb{C}^4.$$

Let $\mathscr{U}_1$ be a second conic open neighborhood of $\varrho^{0\prime}$ such that $\overline{\mathscr{U}_1} \subset \mathscr{U}_0$. Since $\mathbb{S}_{\overline{\mathscr{U}_1}}$ is compact (recall that W is bounded), there exists $C_0 > 0$ such that

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geqslant C_0|\mathbf{z}|_{\mathbb{C}^4}^2, \qquad \mathbf{z} = (z_0, \ldots, z_3) \in \mathbb{C}^4, \ \varrho' \in \mathbb{S}_{\overline{\mathscr{U}_1}}.$$

Introducing the map $N_t\varrho' = (x, t\xi', t\tau, t\sigma)$, for $\varrho' = (x, \xi', \tau, \sigma)$ with $t = |(\xi', \tau, \sigma)|^{-1}$ one has

$$(5.1) \qquad \sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(N_t\varrho', \mathbf{z})\big|^2 \geqslant C_0|\mathbf{z}|_{\mathbb{C}^4}^2, \qquad \mathbf{z} = (z_0, \ldots, z_3) \in \mathbb{C}^4, \ \varrho' \in \mathscr{U}_1,$$

since $N_t \varrho' \in \mathbb{S}_{\overline{\mathscr{U}_1}}$. Now, for $j = 1, 2$ one has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) = \sum_{\ell=0}^{k_j} e_{j,\varrho^{0\prime}}^{\ell}(\varrho') z_{\ell},$$

with $e_{j,\varrho^{0\prime}}^{\ell}(\varrho')$ homogeneous of degree $k_j - \ell$, and for $3 \leqslant j \leqslant m'$ one has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) = \sum_{\ell=0}^{3} e_{j,\varrho^{0\prime}}^{\ell}(\varrho') z_{\ell},$$

with $e_{j,\varrho^{0\prime}}^{\ell}(\varrho')$ homogeneous of degree $m^+ + j - \ell - 3$. We define $\boldsymbol{z}' \in \mathbb{C}^4$ by $z_{\ell}' = t^{\ell - 7/2} z_{\ell}$, $\ell = 0, \dots, 3$. One has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(N_t \varrho', \boldsymbol{z}') = t^{k_j - 7/2} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}), \qquad j = 1, 2,$$

and

$$\Sigma_{e_{j,\varrho^{0\prime}}}(N_t \varrho', \boldsymbol{z}') = t^{m^+ + j - 13/2} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}), \qquad j = 3, \dots, m'.$$

Thus, from (5.1) we deduce

$$(5.2) \quad \sum_{j=1}^{2} \lambda_{\mathsf{T},\tau}^{2(7/2 - k_j)} \left| \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) \right|^2 + \sum_{j=3}^{m'} \lambda_{\mathsf{T},\tau}^{2(13/2 - m^+ - j)} \left| \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \boldsymbol{z}) \right|^2$$
$$\geqslant C_0 \sum_{\ell=0}^{3} \lambda_{\mathsf{T},\tau}^{2(7/2 - \ell)} |z_{\ell}|^2,$$

for $\boldsymbol{z} = (z_0, \dots, z_3) \in \mathbb{C}^4$, and $\varrho' \in \mathscr{U}_1$, since $t \asymp \lambda_{\mathsf{T},\tau}^{-1}$ as $\tau \gtrsim \sigma$ in $\mathscr{U}_1$.

We now choose $\mathscr{U}$ a conic open neighborhood of $\varrho^{0\prime}$, such that $\overline{\mathscr{U}} \subset \mathscr{U}_1$. Let $\chi \in S_{\mathsf{T},\tau}^0$ be as in the statement and let $\widetilde{\chi} \in S_{\mathsf{T},\tau}^0$ be homogeneous of degree 0, with $\mathrm{supp}(\widetilde{\chi}) \subset \mathscr{U}_1$ and $\widetilde{\chi} \equiv 1$ in a neighborhood of $\overline{\mathscr{U}}$, and thus in a neighborhood of $\mathrm{supp}(\chi)$.

For $j = 3, \dots, m'$ one has $e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \kappa_{\varrho^{0\prime}}^{+}(\varrho', \xi_d) \xi_d^{j-3} \in S_{\tau}^{m^+ + j - 3, 0}$. Set $E_j = \mathrm{Op}(\widetilde{\chi} e_{j,\varrho^{0\prime}})$. The introduction of $\widetilde{\chi}$ is made such that $\widetilde{\chi} e_{j,\varrho^{0\prime}}$ is defined on the whole tangential phase-space. Observe that

$$\mathscr{B}(w) = \sum_{j=1}^{2} \left| B_{j,\varphi} w_{|x_d = 0^+} \right|_{7/2 - k_j, \tau}^2 + \sum_{j=3}^{m'} \left| E_j w_{|x_d = 0^+} \right|_{13/2 - m^+ - j, \tau}^2$$
$$= \sum_{j=1}^{2} \left| \Lambda_{\mathsf{T},\tau}^{7/2 - k_j} B_{j,\varphi} w_{|x_d = 0^+} \right|_{\partial}^2 + \sum_{j=3}^{m'} \left| \Lambda_{\mathsf{T},\tau}^{13/2 - m^+ - j} E_j w_{|x_d = 0^+} \right|_{\partial}^2$$

is a boundary quadratic form of type $(3, 1/2)$ as in Definition 2.4. From Proposition 2.6 and (5.2) we have

$$(5.3) \quad |\,\mathrm{tr}(u)|_{3, 1/2, \tau}^2 \lesssim \sum_{j=1}^{2} \left| B_{j,\varphi} u_{|x_d = 0^+} \right|_{7/2 - k_j, \tau}^2 + \sum_{j=3}^{m'} \left| E_j u_{|x_d = 0^+} \right|_{13/2 - m^+ - j, \tau}^2$$
$$+ |\,\mathrm{tr}(v)|_{3, -N, \tau}^2.$$

for $u = \mathrm{Op}_{\mathsf{T}}(\chi) v$ and $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large.

In $\mathscr{U}_1$ one can write

$$p_{\sigma,\varphi} = p^+_{\sigma,\varphi} p^-_{\sigma,\varphi} = \kappa^+_{\varrho^{0\prime}} \kappa^-_{\varrho^{0\prime}},$$

with $\kappa^+_{\varrho^{0\prime}}$ of degree m^+ and $\kappa^-_{\varrho^{0\prime}}$ of degree m^-. In fact we set

$$\widetilde{\kappa}^+_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^+} \big(\xi_d - \widetilde{\chi}\rho_j(\varrho')\big), \qquad \widetilde{\kappa}^-_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^-} \big(\xi_d - \widetilde{\chi}\rho_j(\varrho')\big),$$

with the notation of Section 4.4, thus making the two symbols defined on the whole tangential phase-space. In $\mathscr{U}$, one has also

$$p_{\sigma,\varphi} = \widetilde{\kappa}^+_{\varrho^{0\prime}} \widetilde{\kappa}^-_{\varrho^{0\prime}}.$$

The factor $\widetilde{\kappa}^-_{\varrho^{0\prime}}$ is associated with roots with negative imaginary part. With Lemma A.1 given in Appendix A.1 one has the following microlocal elliptic estimate

$$\|\mathrm{Op}_\tau(\chi)w\|_{m^-,\tau} + |\,\mathrm{tr}(\mathrm{Op}_\tau(\chi)w)|_{m^--1,1/2,\tau} \lesssim \|\mathrm{Op}_\tau(\widetilde{\kappa}^-_{\varrho^{0\prime}})\mathrm{Op}_\tau(\chi)w\|_+ + \|w\|_{m^-,-N,\tau},$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geqslant \kappa_0\tau$ chosen sufficiently large. We apply this inequality to $w = \mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})v$. Since

$$\mathrm{Op}_\tau(\widetilde{\kappa}^-_{\varrho^{0\prime}})\mathrm{Op}_\tau(\chi)\mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}}) = \mathrm{Op}_\tau(\chi)P_{\sigma,\varphi} \mod \Psi^{4,-1}_\tau,$$

one obtains

$$|\,\mathrm{tr}(\mathrm{Op}_\tau(\chi)\mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})v)|_{m^--1,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau}.$$

With $[\mathrm{Op}_\tau(\chi), \mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})] \in \Psi^{m^+,-1}_\tau$ one then has

$$|\,\mathrm{tr}(\mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})u)|_{m^--1,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\,\mathrm{tr}(v)|_{3,-1/2,\tau},$$

with $u = \mathrm{Op}_\tau(\chi)v$ as above, using that $m^+ + m^- = 4$. Note that

$$|\,\mathrm{tr}(\mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})u)|_{m^--1,1/2,\tau} \asymp \sum_{j=0}^{m^--1} |D^j_d \mathrm{Op}_\tau(\widetilde{\kappa}^+_{\varrho^{0\prime}})u_{|x_d=0^+}|_{m^--j-1/2,\tau}$$

$$\gtrsim \sum_{j=3}^{m'} |E_j u_{|x_d=0^+}|_{5/2+m^--j,\tau} - |\,\mathrm{tr}(v)_{|x_d=0^+}|_{3,-1/2,\tau},$$

using that $\xi^j_d \widetilde{\kappa}^+_{\varrho^{0\prime}} = \widetilde{\chi}e_{j+3,\varrho^{0\prime}}$ in a conic neighborhood of $\mathrm{supp}(\chi)$ and using that $m^- = m' - 2$. We thus obtain

$$\sum_{j=3}^{m'} |E_j u_{|x_d=0^+}|_{13/2-m^+-j,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\,\mathrm{tr}(v)|_{3,-1/2,\tau},$$

since $13/2 - m^+ = 5/2 + m^-$. With (5.3) then one finds

$$|\,\mathrm{tr}(u)|_{3,1/2,\tau} \lesssim \sum_{j=1}^{2} |B_{j,\varphi}u_{|x_d=0^+}|^2_{7/2-k_j,\tau} + \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\,\mathrm{tr}(v)|_{3,-1/2,\tau}.$$

In addition, observing that

$$|B_{j,\varphi}u_{|x_d=0^+}|_{7/2-k_j,\tau} \lesssim |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + |\,\mathrm{tr}(v)|_{3,-1/2,\tau},$$

the result of Proposition 5.2 follows. $\qquad\square$

5.2. **Proof of Proposition 5.1.** — As mentioned above the proof relies on a patching procedure of microlocal estimates given by Proposition 5.2.

Let $0 < \kappa_0' < \kappa_0$. We set

$$\Gamma_{+,\kappa_0}^{d-1} = \{(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty) \mid \tau \geqslant \kappa_0 \sigma, \},$$

and

$$\mathbb{S}_{+,\kappa_0}^{d-1} = \{(\xi', \tau, \sigma) \in \Gamma_{+,\kappa_0}^{d-1} \mid |(\xi', \tau, \sigma)| = 1\}.$$

Consider $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{S}_{+,\kappa_0}^{d-1}$. Since the Lopatinskiĭ-Šapiro condition holds at $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$, we can invoke Proposition 5.2:

(1) There exists a conic open neighborhood $\mathscr{U}_{\varrho^{0\prime}}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0' \sigma$;

(2) For any $\chi_{\varrho^{0\prime}} \in S_{\tau,\tau}^0$ homogeneous of degree 0 supported in $\mathscr{U}_{\varrho^{0\prime}}$ the estimate of Proposition 5.2 applies to $\mathrm{Op}_\tau(\chi_{\varrho^{0\prime}})v$ for $\tau \geqslant \max(\tau_{\varrho^{0\prime}}, \kappa_0 \sigma)$.

Without any loss of generality we may choose $\mathscr{U}_{\varrho^{0\prime}}$ of the form $\mathscr{U}_{\varrho^{0\prime}} = \mathscr{O}_{\varrho^{0\prime}} \times \Gamma_{\varrho^{0\prime}}$, with $\mathscr{O}_{\varrho^{0\prime}} \subset W$ an open neighborhood of x^0 and $\Gamma_{\varrho^{0\prime}}$ a conic open neighborhood of $(\xi^{0\prime}, \tau^0, \sigma^0)$ in $\mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0' \sigma$.

Since $\{x^0\} \times \mathbb{S}_{+,\kappa_0}^{d-1}$ is compact we can extract a finite covering of it by open sets of the form of $\mathscr{U}_{\varrho^{0\prime}}$. We denote by $\widetilde{\mathscr{U}_i}$, $i \in I$ with $|I| < \infty$, such a finite covering. This is also a finite covering of $\{x^0\} \times \Gamma_{+,\kappa_0}^{d-1}$.

Each $\widetilde{\mathscr{U}_i}$ has the form $\widetilde{\mathscr{U}_i} = \mathscr{O}_i \times \Gamma_i$, with $\mathscr{O}_i$ an open neighborhood of x^0 and Γ_i a conic open set in $\mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0' \sigma$.

We set $\mathscr{O} = \cap_{i \in I} \mathscr{O}_i$ and $\mathscr{U}_i = \mathscr{O} \times \Gamma_i$, $i \in I$. Let W^0 be an open neighborhood of x^0 such that $W^0 \Subset \mathscr{O}$. The open sets $\mathscr{U}_i$ give also an open covering of $\overline{W^0} \times \mathbb{S}_{+,\kappa_0}^{d-1}$ and $\overline{W^0} \times \Gamma_{+,\kappa_0}^{d-1}$. With this second covering we associate a partition of unity χ_i, $i \in I$, of $\overline{W^0} \times \mathbb{S}_{+,\kappa_0}^{d-1}$, where each χ_i is chosen smooth and homogeneous of degree one for $|(\xi', \tau, \sigma)| \geqslant 1$, that is:

$$\sum_{i \in I} \chi_i(\varrho') = 1$$

for $\varrho' = (x, \xi', \tau, \sigma)$ in a neighborhood of $\overline{W^0} \times \Gamma_{+,\kappa_0}^{d-1}$, and $|(\xi', \tau, \sigma)| \geqslant 1$.

Let $u \in \overline{\mathscr{C}_c^\infty}(W_+^0))$. Since each χ_i is in $S_{\tau,\tau}^0$ and supported in $\mathscr{U}_i$, Proposition 5.2 applies:

$$(5.4) \quad |\,\mathrm{tr}(\mathrm{Op}_\tau(\chi_i)v)|_{3,1/2,\tau} \leqslant C_i \Big(\sum_{j=1}^2 |B_{j,\varphi} v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi} v\|_+$$
$$+ \|v\|_{4,-1,\tau} + |\,\mathrm{tr}(v)|_{3,-1/2,\tau} \Big),$$

for some $C_i > 0$, for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_i, \kappa_0 \sigma)$ for some $\tau_i > 0$.

We set $\widetilde{\chi} = 1 - \sum_{i \in I} \chi_i$. One has $\widetilde{\chi} \in S_{\tau,\tau}^{-\infty}$ microlocally in a neighborhood of $\overline{W^0} \times \Gamma_{+,\kappa_0}^{d-1}$. Thus, considering the definition of $\Gamma_{+,\kappa_0}^{d-1}$, if one imposes $\tau \geqslant \kappa_0 \sigma$, as we do, then $\widetilde{\chi} \in S_{\tau,\tau}^{-\infty}$ locally in a neighborhood of $\overline{W^0}$.

For any $N \in \mathbb{N}$ using that $\mathrm{supp}(v) \subset W^0$ one has

$$
\begin{aligned}
|\operatorname{tr}(v)|_{3,1/2,\tau} &\leqslant \sum_{i \in I} |\operatorname{tr}(\mathrm{Op}_\tau(\chi_i)v)|_{3,1/2,\tau} + |\operatorname{tr}(\mathrm{Op}_\tau(\widetilde{\chi})v)|_{3,1/2,\tau} \\
&\lesssim \sum_{i \in I} |\operatorname{tr}(\mathrm{Op}_\tau(\chi_i)v)|_{3,1/2,\tau} + |\operatorname{tr}(v)|_{3,-N,\tau} \\
&\lesssim \sum_{i \in I} |\operatorname{tr}(\mathrm{Op}_\tau(\chi_i)v)|_{3,1/2,\tau} + \|v\|_{4,-N,\tau}.
\end{aligned}
$$

Summing estimates (5.4) together for $i \in I$ we thus obtain

$$
|\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \sum_{j=1}^2 |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau},
$$

for $\tau \geqslant \max(\max_i \tau_i, \kappa_0 \sigma)$. Therefore, by choosing $\tau \geqslant \kappa_0 \sigma$ sufficiently large one obtains the result of Proposition 5.1. □

6. Microlocal estimates for second-order factors

We recall that $P_\sigma = \Delta^2 - \sigma^4 = (-\Delta - \sigma^2)(-\Delta + \sigma^2)$ with $\sigma \geqslant 0$. Set $Q_\sigma^j = -\Delta + (-1)^j \sigma^2$; then $P_\sigma = Q_\sigma^1 Q_\sigma^2$. We also set $Q = -\Delta$, that is, $Q = Q_0^1 = Q_0^2$.

The principal symbols of Q_σ^j and Q are given by

$$
(6.1) \qquad q_\sigma^j(x,\xi) = \xi_d^2 + r(x,\xi') + (-1)^j \sigma^2 \quad \text{and} \quad q(x,\xi) = \xi_d^2 + r(x,\xi'),
$$

respectively. The conjugated operator $P_{\sigma,\varphi} = e^{\tau\varphi} P_\sigma e^{-\tau\varphi}$ reads

$$
P_{\sigma,\varphi} = Q_{\sigma,\varphi}^1 Q_{\sigma,\varphi}^2, \quad \text{with} \quad Q_{\sigma,\varphi}^j = e^{\tau\varphi} Q_\sigma^j e^{-\tau\varphi}.
$$

We set

$$
Q_s^j = \frac{Q_{\sigma,\varphi}^j + (Q_{\sigma,\varphi}^j)^*}{2} \quad \text{and} \quad Q_a = \frac{Q_{\sigma,\varphi}^j - (Q_{\sigma,\varphi}^j)^*}{2i},
$$

both formally selfadjoint and such that $Q_{\sigma,\varphi}^j = Q_s^j + iQ_a$. Note that Q_a is independent of σ. Their respective principal symbols are

$$
\begin{aligned}
q_s^j(x,\xi,\tau,\sigma) &= \xi_d^2 - (\tau\partial_d\varphi)^2 + r(x,\xi') + (-1)^j \sigma^2 - \tau^2 r(x, d_{x'}\varphi), \\
q_a(x,\xi,\tau) &= 2\tau\xi_d \partial_d\varphi + 2\tau\widetilde{r}(x,\xi', d_{x'}\varphi).
\end{aligned}
$$

Note that Q_s^j and Q_a take the forms

$$
(6.2) \qquad Q_s^j = D_d^2 + T_s^j, \qquad Q_a = \tau(\partial_d\varphi D_d + D_d \partial_d\varphi) + T_a,
$$

where $T_s^j, T_a \in \mathscr{D}_{\tau,\tau}^2$ are such that $(T_s^j)^* = T_s^j$ and $T_a^* = T_a$. Naturally, the principal symbol of $Q_{\sigma,\varphi}^j$ is

$$
q_{\sigma,\varphi}^j(x,\xi,\tau) = q_s^j(x,\xi,\tau,\sigma) + iq_a(x,\xi,\tau).
$$

The principal symbol of $Q_{\sigma,\varphi}^j = e^{\tau\varphi} Q_\sigma^j e^{-\tau\varphi}$ is

$$
q_{\sigma,\varphi}^j(x,\xi,\tau,\sigma) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi') + (-1)^j \sigma^2 - \tau^2 r(x, d_{x'}\varphi) + 2i\tau\widetilde{r}(x,\xi', d_{x'}\varphi).
$$

As in Section 4.2 we let $\alpha_j \in \mathbb{C}$ be such that

$$\alpha_j(x, \xi', \tau, \sigma)^2 = r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j \sigma^2$$
$$= r(x, \xi') - \tau^2 r(x, d_{x'}\varphi) + 2i\tau\widetilde{r}(x, \xi', d_{x'}\varphi) + (-1)^j \sigma^2,$$

and $\mathrm{Re}\,\alpha_j \geqslant 0$. Note that uniqueness in the choice of α_j holds except if $r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j \sigma^2 \in \mathbb{R}^-$; this lack of uniqueness in that case is however not an issue in what follows. One has

$$q_{\sigma,\varphi}^j(x, \xi', \xi_d, \tau) = (\xi_d + i\tau\partial_d\varphi)^2 + \alpha_j(x, \xi', \tau, \sigma)^2$$
$$= \big(\xi_d + i\tau\partial_d\varphi + i\alpha_j(x, \xi', \tau, \sigma)\big)\big(\xi_d + i\tau\partial_d\varphi - i\alpha_j(x, \xi', \tau, \sigma)\big).$$

We recall from (4.4)–(4.5) that we write $q_{\sigma,\varphi}^j(x, \xi', \xi_d, \tau) = (\xi_d - \pi_{j,1})(\xi_d - \pi_{j,2})$ with

$$\pi_{j,1} = -i\tau\partial_d\varphi - i\alpha_j(x, \xi', \tau, \sigma) \quad \text{and} \quad \pi_{j,2} = -i\tau\partial_d\varphi + i\alpha_j(x, \xi', \tau, \sigma).$$

The roots $\pi_{j,k}$, $k = 1, 2$ are functions of x, ξ', τ and σ.

We denote by B a boundary operator of order k that takes the form

$$B(x, D) = B^k(x, D') + B^{k-1}(x, D')D_d,$$

with $B^k(x, D')$ and $B^{k-1}(x, D')$ tangential differential operators of order k and $k-1$ respectively. The boundary operator $B(x, D)$ has $b(x, \xi) = b^k(x, \xi') + b^{k-1}(x, \xi')\xi_d$ for principal symbol. The conjugate boundary operator $B_\varphi = e^{\tau\varphi}Be^{-\tau\varphi}$ is then given by

$$B_\varphi(x, D, \tau) = B_\varphi^k(x, D', \tau) + B_\varphi^{k-1}(x, D', \tau)(D_d + i\tau\partial_d\varphi)$$
$$= \widehat{B}_\varphi^k(x, D', \tau) + B_\varphi^{k-1}(x, D', \tau)D_d,$$

with $\widehat{B}_\varphi^k(x, D', \tau) = B_\varphi^k(x, D', \tau) + i\tau B_\varphi^{k-1}(x, D', \tau)\partial_d\varphi$. The principal symbol of $B_\varphi(x, D, \tau)$ is

$$b_\varphi(x, \xi, \tau) = \widehat{b}_\varphi^k(x, \xi', \tau) + b_\varphi^{k-1}(x, \xi', \tau)\xi_d,$$

where $b_\varphi^{k-1}(x, \xi', \tau)$ is homogeneous of degree $k-1$ in $\lambda_{\tau,\tau}$ and

$$\widehat{b}_\varphi^k(x, \xi', \tau) = b_\varphi^k(x, \xi', \tau) + i\tau b_\varphi^{k-1}(x, \xi', \tau)\partial_d\varphi$$

is homogeneous of degree k in $\lambda_{\tau,\tau}$.

6.1. Sub-ellipticity. — Set

$$q_s(x, \xi, \tau) = \xi_d^2 + r(x, \xi') - (\tau\partial_d\varphi)^2 - r(x, \tau d_{x'}\varphi) = |\xi|_x^2 - |\tau d\varphi|_x^2,$$

where $|\xi|_x^2 = \xi_d^2 + r(x, \xi')$. One has $q_s^j = q_s + (-1)^j\sigma^2$. Observe that $\{q_s^j, q_a\} = \{q_s, q_a\}$.

Definition 6.1 (Sub-ellipticity). — Let W be a bounded open subset of $\mathbb{R}^d$ and $\varphi \in \mathscr{C}^\infty(\overline{W})$ such that $|d_x\varphi| > 0$. Let $j = 1$ or 2. We say that the couple (Q_σ^j, φ) satisfies the sub-ellipticity condition in $\overline{W}$ if there exist $C > 0$ and $\tau_0 > 0$ such that for $\sigma > 0$

$$\forall\, (x, \xi) \in \overline{W} \times \mathbb{R}^d,\ \tau \geqslant \tau_0\sigma,$$
$$q_{\sigma,\varphi}^j(x, \xi, \tau) = 0 \implies \{q_s^j, q_a\}(x, \xi, \tau) = \{q_s, q_a\}(x, \xi, \tau) \geqslant C > 0.$$

Remark 6.2. — Note that with homogeneity the sub-ellipticity property also reads

$$\forall\, (x, \xi) \in \overline{W} \times \mathbb{R}^d,\ \tau \geqslant \tau_0\sigma,\ q_{\sigma,\varphi}^j(x, \xi, \tau) = 0 \implies \{q_s^j, q_a\}(x, \xi, \tau) \geqslant C\lambda_\tau^3.$$

Proposition 6.3. — *Let W be a bounded open subset of $\mathbb{R}^d$ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $\psi \geqslant 0$ and $|d_x\psi| \geqslant C > 0$ on $\overline{W}$. Let $\tau_0 > 0$. Then, there exists $\gamma_0 \geqslant 1$ such that (Q^j_σ, φ) satisfies the sub-ellipticity condition on $\overline{W}$ for $\tau \geqslant \tau_0 \sigma$ for $\varphi = e^{\gamma\psi}$, with $\gamma \geqslant \gamma_0$, for both $j = 1$ and 2.*

Proof. — We note that $|d_x\varphi(x)| \neq 0$. The proof is slightly different whether one considers the symbol $q^1_{\sigma,\varphi}$ or the symbol $q^2_{\sigma,\varphi}$.

Case 1: proof for $q^1_{\sigma,\varphi}$. — Assume that $q^1_{\sigma,\varphi} = 0$. Thus $|\xi|^2_x - |\tau d\varphi|^2_x - \sigma^2 = 0$ implying $|\xi| \sim \sigma + \gamma\tau\varphi$. On the one hand by Lemma 3.55 in [31], one has

$$(6.3) \quad \{q_s, q_a\}(x, \xi, \tau) = \tau(\gamma^2\varphi)(\gamma\varphi)^2 \left((H_q\psi(x, \beta))^2 + 4\tau^2 q(x, d\psi(x))^2 \right)$$
$$+ (\gamma\varphi)^3 \frac{1}{2i}\{\overline{q_\psi}, q_\psi\}(x, \beta, \tau),$$

with $\beta = \xi/(\gamma\varphi)$, and where H_q denotes the Hamiltonian vector field associated with the symbol q as defined in (6.1). Here, q_ψ denotes the principal symbol of $e^{\tau\psi}Qe^{-\tau\psi}$, that is,

$$q_\psi(x, \xi, \tau) = q(x, \xi + i\tau d_x\psi(x))$$
$$= (\xi_d + i\tau\partial_d\psi(x))^2 + r(x, \xi' + i\tau d_{x'}\psi(x)).$$

On the other hand, one has $(H_q\psi(x, \beta))^2 + 4\tau^2 q(x, d\psi(x))^2 \gtrsim \tau^2$ and since $\frac{1}{2i}\{\overline{q_\psi}, q_\psi\}(x, \beta, \tau)$ is homogeneous of degree 3 in (β, τ), we obtain

$$\{q_s, q_a\} \geqslant C\gamma(\gamma\tau\varphi)^3 - C'(\gamma\tau\varphi + |\beta|\gamma\varphi)^3 = C\gamma\widetilde{\tau}^3 - C''(\widetilde{\tau} + |\xi|)^3,$$

with $\widetilde{\tau} = \gamma\tau\varphi$. Yet, one has $|\xi| \sim \sigma + \widetilde{\tau}$ implying

$$\{q_s, q_a\} \geqslant C\gamma\widetilde{\tau}^3 - C'''(\widetilde{\tau} + |\xi|)^3 \geqslant C\gamma\widetilde{\tau}^3 - C^{(4)}(\widetilde{\tau} + \sigma)^3.$$

Since $\psi \geqslant 0$ and $\gamma \geqslant 1$ one has $\varphi \geqslant 1$ implying $\tau_0\sigma \leqslant \tau \lesssim \widetilde{\tau}$ and thus

$$\{q_s, q_a\}(x, \xi, \tau) \geqslant \widetilde{\tau}^3(C\gamma - C^{(5)}).$$

It follows that for γ chosen sufficiently large one finds $\{q_s, q_a\}(x, \xi, \tau) \geqslant C > 0$.

Case 2: proof for $q^2_{\sigma,\varphi}$. — Assume that $q^2_{\sigma,\varphi} = 0$. Then $|\xi|^2_x + \sigma^2 = |\tau d\varphi|^2_x$ implying $|\xi| + \sigma \sim \tau|d\varphi| \sim \widetilde{\tau}$. The same computation as in Case 1 gives

$$\{q_s, q_a\}(x, \xi, \tau) \geqslant C\gamma\widetilde{\tau}^3 - C'(\widetilde{\tau} + |\xi|)^3.$$

Here $|\xi| + \widetilde{\tau} \lesssim \widetilde{\tau}$ yielding

$$\{q_s, q_a\}(x, \xi, \tau) \geqslant (C\gamma - C'')\widetilde{\tau}^3.$$

The remaining part of the proof is the same. $\square$

Lemma 6.4. — *Let $j = 1$ or 2. Let (Q^j_σ, φ) have the sub-ellipticity property of Definition 6.1 in $\overline{W}$. For $\mu > 0$ one sets $t(\varrho) = \mu((q^j_s)^2 + q^2_a)(\varrho) + \tau\{q^j_s, q_a\}(\varrho)$ with $\varrho = (x, \xi, \tau, \sigma) \in \overline{W} \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$. Let $\tau_0 > 0$. Then, for μ chosen sufficiently large and $\tau \geqslant \tau_0\sigma$ one has $t(\varrho) \geqslant C\lambda^4_\tau$ for some $C > 0$.*

The proof of Lemma 6.4 uses the following lemma.

Lemma 6.5. — *Consider two continuous functions, f and g, defined in a compact set $\mathscr{L}$, and assume that $f \geqslant 0$ and moreover*

$$f(y) = 0 \implies g(y) > 0 \text{ for all } y \in \mathscr{L}.$$

Setting $h_\mu = \mu f + g$, we have $h_\mu \geqslant C > 0$ for $\mu > 0$ chosen sufficiently large.

Proof of Lemma 6.4. — Consider the compact set

$$\mathscr{L} = \{(x, \xi, \tau, \sigma) \mid x \in \overline{W}, \ |\xi|^2 + \tau^2 + \sigma^2 = 1, \ \tau \geqslant \tau_0 \sigma\}.$$

Applying the result of Lemma 6.5 to $t(\varrho)$ on $\mathscr{L}$ with $f = (q_s^j)^2 + q_a^2$ and $g = \tau\{q_s^j, q_a\}$ we find for $t(\varrho) \geqslant C$ on $\mathscr{L}$ for some $C > 0$ for μ chosen sufficiently large. Since $t(\varrho)$ is homogeneous of degree 4 in the variables (ξ, τ, σ) it follows that

$$t(\varrho) \geqslant C(\sigma^2 + \tau^2 + |\xi|^2)^2 \gtrsim \lambda_\tau^4. \qquad \square$$

6.2. Lopatinskiĭ-Šapiro condition for the second-order factors. — Above, in Section 4, the Lopatinskiĭ-Šapiro condition is addressed for the fourth-order operator $P_{\sigma,\varphi}$. Here, we consider the two second-order factors $Q_{\sigma,\varphi}^j$.

With the roots $\pi_{j,1}$ and $\pi_{j,2}$ defined in (4.4)–(4.5) one sets

$$q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau) = \prod_{\substack{\operatorname{Im} \pi_{j,k} \geqslant 0 \\ k=1,2}} \left(\xi_d - \pi_{j,k}(x, \xi', \tau, \sigma)\right).$$

Definition 6.6. — Let $j = 1, 2$. Let $x \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$. Let $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ with $(\xi', \tau, \sigma) \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for (Q_σ^j, B, φ) at $\varrho' = (x, \xi', \tau, \sigma)$ if for any polynomial $f(\xi_d)$ with complex coefficients there exist $c \in \mathbb{C}$ and a polynomial $\ell(\xi_d)$ with complex coefficients such that, for all $\xi_d \in \mathbb{C}$

$$(6.4) \qquad f(\xi_d) = cb_\varphi(x, \xi', \xi_d, \tau) + \ell(\xi_d)q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau).$$

Remark 6.7. — With the Euclidean division of polynomials, we see that it suffices to consider the polynomial $f(\xi_d)$ to be of degree less than that of $q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau)$ in (6.4). Thus, in any case, the degree of $f(\xi_d)$ can be chosen less than or equal to one.

Lemma 6.8. — *Let $j = 1$ or 2. Let $x \in \partial\Omega$ and $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ with $(\xi', \tau, \sigma) \neq 0$. The Lopatinskiĭ-Šapiro condition holds for (Q_σ^j, B, φ) at (x, ξ', τ, σ) if and only if*
  (1) *either $q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau) = 1$;*
  (2) *or $q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau) = \xi_d - \pi$ and $b_\varphi(x, \xi', \pi, \tau) \neq 0$.*

Proof. — If $q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau) = \left(\xi_d - \pi_{j,1}(x, \xi', \tau, \sigma)\right)\left(\xi_d - \pi_{j,2}(x, \xi', \tau, \sigma)\right)$, that is, both roots $\pi_{j,1}(x, \xi', \tau, \sigma)$ and $\pi_{j,2}(x, \xi', \tau, \sigma)$ are in the upper complex half-plane, then condition (6.4) cannot hold, since by Remark 6.7 it means that the vector space of polynomials of degree less than or equal to one would be generated by the single polynomial $b_\varphi(x, \xi', \xi_d, \tau)$.

Suppose that $q_{\sigma,\varphi}^{j,+}(x, \xi', \xi_d, \tau) = \xi_d - \pi$ that is one the root $\pi_{j,1}(x, \xi', \tau, \sigma)$ and $\pi_{j,2}(x, \xi', \tau, \sigma)$ has a nonnegative imaginary part and the other root has a negative

imaginary part. Then, the Lopatinskiĭ-Šapiro condition holds at (x, ξ', σ, τ) if for any $f(\xi_d)$, the polynomial $\xi_d \mapsto f(\xi_d) - c b_\varphi(x, \xi', \xi_d, \tau)$ admits π as a root for some $c \in \mathbb{C}$. A necessary and sufficient condition is then $b_\varphi(x, \xi', \xi_d = \pi, \tau) \neq 0$.

Finally if $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = 1$, that is, both roots $\pi_{j,1}(x, \xi', \tau, \sigma)$ and $\pi_{j,2}(x, \xi', \tau, \sigma)$ lie in the lower complex half-plane, then condition (6.4) trivially holds.   □

6.3. Microlocal estimates. — Here, for $j = 1$ or 2, we establish estimates for the operator Q^j_σ in a microlocal neighborhood of point at the boundary where (Q^j_σ, B, φ) satisfies the Lopatinskiĭ-Šapiro condition (after conjugation) of Definition 6.6.

The quality of the estimation depends on the position of the roots. We shall assume that $\partial_d \varphi > 0$. Thus, from the form of the roots $\pi_{j,1}$ and $\pi_{j,2}$ given in (4.4)–(4.5), the root $\pi_{j,1}$ always lies in the lower complex half-plane. The sign of $\operatorname{Im} \pi_{j,2}$ may however vary. Three cases can thus occur:

(1) The root $\pi_{j,2}$ at the considered point lies in the upper complex half-plane.
(2) The root $\pi_{j,2}$ at the considered point is real.
(3) The root $\pi_{j,2}$ at the considered point lies in the lower complex half-plane.

Proposition 6.9. — *Let $j = 1$ or 2 and $\kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geqslant C > 0$ in W and such that (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\overline{W}$. Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ with $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ nonvanishing with $\tau^0 \geqslant \kappa_1 \sigma^0$ and such that (Q^j_σ, B, φ) satisfies the Lopatinskiĭ-Šapiro condition of Definition 6.6 at $\varrho^{0\prime}$.*

(1) Assume that $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) > 0$. Then, there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0 \sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$(6.5) \quad \|\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)v)|_{1,1/2,\tau}$$
$$\leqslant C\big(\|Q^j_{\sigma,\varphi}v\|_+ + |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}\big),$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$ and $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

(2) Assume that $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) = 0$. Then, there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0 \sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$(6.6) \quad \tau^{-1/2}\|\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)v)|_{1,1/2,\tau}$$
$$\leqslant C\big(\|Q^j_{\sigma,\varphi}v\|_+ + |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}\big),$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$ and $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

(3) Assume that $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) < 0$. Then, there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0 \sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$(6.7) \quad \|\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)v)|_{1,1/2,\tau} \leqslant C\big(\|Q^j_{\sigma,\varphi}v\|_+ + \|v\|_{2,-1,\tau}\big),$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$ and $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

The notation of the function space $\overline{\mathscr{C}}_c^\infty(W_+)$ is introduced in (1.6).

6.3.1. *Case (i): one root lying in the upper complex half-plane.* — One has

$$\operatorname{Im}\pi_{j,2}(\varrho^{0\prime}) > 0 \quad\text{and}\quad \operatorname{Im}\pi_{j,1}(\varrho^{0\prime}) < 0.$$

Since the Lopatinskiĭ-Šapiro condition holds for (Q_σ^j, B, φ) at $\varrho^{0\prime}$, by Lemma 6.8 one has

$$b_\varphi\big(x^0, \xi^{0\prime}, \xi_d = \pi_{j,2}(\varrho^{0\prime}), \tau^0\big) = b\big(x^0, \xi^{0\prime} + i\tau^0 d_{x'}\varphi(x^0), i\alpha_j(\varrho^{0\prime})\big) \neq 0.$$

As the roots $\pi_{j,1}$ and $\pi_{j,2}$ are locally smooth with respect to $\varrho' = (x, \xi', \tau, \sigma)$ and homogeneous of degree one in (ξ', τ, σ), there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_1 > 0$, $C_2 > 0$ such that

$$\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}} \mid |\xi'|^2 + \tau^2 + \sigma^2 = 1\}$$

is compact and

$$\tau \geqslant \kappa_0\sigma, \quad \operatorname{Im}\pi_{j,2}(\varrho') \geqslant C_2\lambda_{\tau,\tau}, \quad\text{and}\quad \operatorname{Im}\pi_{j,1}(\varrho') \leqslant -C_1\lambda_{\tau,\tau},$$

and

$$(6.8) \qquad\qquad b_\varphi(x, \xi', \xi_d = \pi_{j,2}(\varrho'), \tau) \neq 0,$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

We let $\chi \in S_{\tau,\tau}^0$ and $\widetilde{\chi} \in S_{\tau,\tau}^0$ be homogeneous of degree zero in the variable (ξ', τ, σ) and be such that $\operatorname{supp}(\widetilde{\chi}) \subset \mathscr{U}$ and $\widetilde{\chi} \equiv 1$ on a neighborhood of $\operatorname{supp}(\chi)$. From the smoothness and the homogeneity of the roots, one has $\widetilde{\chi}\pi_{j,k} \in S_{\tau,\tau}^1$, $k = 1, 2$. We set

$$L_2 = D_d - \operatorname{Op}_\tau(\widetilde{\chi}\pi_{j,2}) \quad\text{and}\quad L_1 = D_d - \operatorname{Op}_\tau(\widetilde{\chi}\pi_{j,1}).$$

The proof of Estimate (6.5) is based on three lemmas that we now list. Their proofs are given at the end of this section.

The following lemma provides an estimate for L_2 and boundary traces.

Lemma 6.10. — *There exist $C > 0$ and $\tau_0 > 0$ such that for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that*

$$\big|\operatorname{tr}\big(\operatorname{Op}_\tau(\chi)w\big)\big|_{1,1/2,\tau} \leqslant C\big(\big|B_\varphi\operatorname{Op}_\tau(\chi)w_{|x_d=0^+}\big|_{3/2-k,\tau}$$
$$+ \big|L_2\operatorname{Op}_\tau(\chi)w_{|x_d=0^+}\big|_{1/2,\tau}\big) + C_N\big|\operatorname{tr}(w)\big|_{1,-N,\tau},$$

for $\tau \geqslant \max(\tau_0, \kappa_0\sigma)$ and $w \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$.

The proof of Lemma 6.10 relies on the Lopatinskiĭ-Šapiro condition. The following lemma gives an estimate for L_1.

Lemma 6.11. — *Let $\chi \in S_{\tau,\tau}^0$, homogeneous of degree 0, be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$ and $s \in \mathbb{R}$. There exist $C > 0$, $\tau_0 > 0$ and $N \in \mathbb{N}$ such that*

$$\|\operatorname{Op}_\tau(\chi)w\|_{1,s,\tau} + |\operatorname{Op}_\tau(\chi)w_{|x_d=0^+}|_{s+1/2,\tau} \leqslant C\big(\|L_1\operatorname{Op}_\tau(\chi)w\|_{0,s,\tau} + \|w\|_{0,-N,\tau}\big),$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$ and $\tau \geqslant \max(\tau_0, \kappa_0\sigma)$.

The proof of Lemma 6.11 is based on a multiplier method and relies on the fact that the root $\pi_{j,1}$ that appears in the principal symbol of L_1 lies in the lower complex half-plane.

The following lemma gives an estimate for L_2.

Lemma 6.12. — *Let $\chi \in S_{\tau,\tau}^0$, homogeneous of degree 0, be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$ and $s \in \mathbb{R}$. There exist $C > 0$, $\tau_0 > 0$ and $N \in \mathbb{N}$ such that*

$$\|\operatorname{Op}_\tau(\chi)w\|_{1,s,\tau} \leqslant C\big(\|L_2\operatorname{Op}_\tau(\chi)w\|_{0,s,\tau} + |\operatorname{Op}_\tau(\chi)w_{|x_d=0^+}|_{s+1/2,\tau} + \|w\|_{0,-N,\tau}\big),$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$ and $\tau \geqslant \max(\tau_0, \kappa_0\sigma)$.

Note that this estimate is weaker than that of Lemma 6.11. Observing that

$$L_1\operatorname{Op}_\tau(\chi)L_2 = \operatorname{Op}_\tau(\chi)L_1L_2 \mod \Psi_\tau^{1,0}$$
$$= \operatorname{Op}_\tau(\chi)Q_{\sigma,\varphi}^j \mod \Psi_\tau^{1,0},$$

and applying Lemma 6.11 to $w = L_2v$ with $s = 0$, one obtains

$$\|\operatorname{Op}_\tau(\chi)L_2v\|_{1,\tau} + |\operatorname{Op}_\tau(\chi)L_2v_{|x_d=0^+}|_{1/2,\tau} \lesssim \|L_1\operatorname{Op}_\tau(\chi)L_2v\|_+ + \|v\|_{1,-N,\tau}$$
$$\lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{1,\tau},$$

for $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large. We set $u = \operatorname{Op}_\tau(\chi)v$, and using the trace inequality

$$|w_{|x_d=0^+}|_{s,\tau} \lesssim \|w\|_{s+1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d) \text{ and } s > 0,$$

we have

$$\|L_2u\|_{1,\tau} + |L_2u_{|x_d=0^+}|_{1/2,\tau} \lesssim \|\operatorname{Op}_\tau(\chi)L_2v\|_{1,\tau} + |\operatorname{Op}_\tau(\chi)L_2v_{|x_d=0^+}|_{1/2,\tau}$$
$$+ \|v\|_{1,\tau} + |v_{|x_d=0^+}|_{1/2,\tau}$$
$$\lesssim \|\operatorname{Op}_\tau(\chi)L_2v\|_{1,\tau} + |\operatorname{Op}_\tau(\chi)L_2v_{|x_d=0^+}|_{1/2,\tau} + \|v\|_{1,\tau}.$$

Therefore, we obtain

$$\|L_2u\|_{1,\tau} + |L_2u_{|x_d=0^+}|_{1/2,\tau} \lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{1,\tau}.$$

With Lemma 6.10, one has the estimate

$$\big|\operatorname{tr}(u)\big|_{1,1/2,\tau} + \|L_2u\|_{1,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{2,-1,\tau},$$

for $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large using the following trace inequality

$$\big|\operatorname{tr}(w)\big|_{m,s,\tau} \lesssim \|w\|_{m+1,s-1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d) \text{ and } m \in \mathbb{N}, \ s \in \mathbb{R}.$$

With Lemma 6.12 for $s = 1$ one obtains

$$\|u\|_{1,1,\tau} + \big|\operatorname{tr}(u)\big|_{1,1/2,\tau} + \|L_2u\|_{1,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{2,-1,\tau},$$

for $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large. Finally, we write

$$\|D_d u\|_{1,\tau} \leqslant \|L_2u\|_{1,\tau} + \|\operatorname{Op}_\tau(\widetilde{\chi}\pi_{j,2})u\|_{1,\tau} \lesssim \|L_2u\|_{1,\tau} + \|u\|_{1,1,\tau},$$

yielding

$$\|u\|_{2,\tau} + \big|\operatorname{tr}(u)\big|_{1,1/2,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{2,-1,\tau}.$$

As $u = \mathrm{Op}_\tau(\chi)v$, with a commutator argument we obtain

$$
\begin{aligned}
|B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} &\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + |\operatorname{tr}(v)|_{1,-1/2,\tau} \\
&\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}.
\end{aligned}
$$

yielding (6.5) and thus concluding the proof of Proposition 6.9 in Case (i). $\qquad\square$

We now provide the proofs the three key lemmas used above.

Proof of Lemma 6.10. — Set

$$
\begin{aligned}
\mathcal{T}(w) &= |B_\varphi w_{|x_d=0^+}|_{3/2-k,\tau}^2 + |L_2 w_{|x_d=0^+}|_{1/2,\tau}^2 \\
&= |\Lambda_{\mathsf{T},\tau}^{3/2-k} B_\varphi w_{|x_d=0^+}|_\partial^2 + |\Lambda_{\mathsf{T},\tau}^{1/2} L_2 w_{|x_d=0^+}|_\partial^2.
\end{aligned}
$$

This is a boundary differential quadratic form of type $(1,1/2)$ in the sense of Definition 2.4. The associated bilinear symbol is given by

$$
\begin{aligned}
\Sigma_{\mathcal{T}}(\varrho', \boldsymbol{z}, \boldsymbol{z}') = \lambda_{\mathsf{T},\tau}^{3-2k}\Big(& \widehat{b}_\varphi^k(x,\xi',\tau)z_0 \\
&+ b_\varphi^{k-1}(x,\xi',\tau)z_1\Big)\Big(\overline{\widehat{b}_\varphi^k}(x,\xi',\tau)\overline{z}_0' + \overline{b_\varphi^{k-1}}(x,\xi',\tau)\overline{z}_1'\Big) \\
&+ \lambda_{\mathsf{T},\tau}\big(z_1 - \widetilde{\chi}\pi_{j,2}(\varrho')z_0\big)\big(\overline{z}_1' - \widetilde{\chi}\pi_{j,2}(\varrho')\overline{z}_0'\big),
\end{aligned}
$$

with $\boldsymbol{z} = (z_0, z_1) \in \mathbb{C}^2$ and $\boldsymbol{z}' = (z_0', z_1') \in \mathbb{C}^2$, yielding

$$
\Sigma_{\mathcal{T}}(\varrho', \boldsymbol{z}, \boldsymbol{z}) = \lambda_{\mathsf{T},\tau}^{3-2k}\big|\widehat{b}_\varphi^k(x,\xi',\tau)z_0 + b_\varphi^{k-1}(x,\xi',\tau)z_1\big|^2 + \lambda_{\mathsf{T},\tau}\big|z_1 - \widetilde{\chi}\pi_{j,2}(\varrho')z_0\big|^2.
$$

One has $\Sigma_{\mathcal{T}}(\varrho', \boldsymbol{z}, \boldsymbol{z}) \geqslant 0$. For $\boldsymbol{z} \neq (0,0)$ if $\Sigma_{\mathcal{T}}(\varrho', \boldsymbol{z}, \boldsymbol{z}) = 0$ then

$$
\begin{cases}
z_1 = \widetilde{\chi}\pi_{j,2}(\varrho')z_0, \\
\widehat{b}_\varphi^k(x,\xi',\tau)z_0 + b_\varphi^{k-1}(x,\xi',\tau)z_1 = 0,
\end{cases}
$$

implying that $z_0 \neq 0$ and

$$
b_\varphi\big(x,\xi',\xi_d = \widetilde{\chi}\pi_{j,2}(\varrho'),\tau\big) = \widehat{b}_\varphi^k(x,\xi',\tau) + b_\varphi^{k-1}(x,\xi',\tau)\widetilde{\chi}\pi_{j,2}(\varrho') = 0.
$$

Let $\mathscr{U}_1 \subset \mathscr{U}$ be a conic open set such that $\operatorname{supp}(\chi) \subset \mathscr{U}_1$ and $\widetilde{\chi} = 1$ in a conic neighborhood of $\overline{\mathscr{U}_1}$. Then, for $\varrho' \in \overline{\mathscr{U}_1}$ one has

$$
b_\varphi\big(x,\xi',\xi_d = \widetilde{\chi}\pi_{j,2}(\varrho'),\tau\big) = b_\varphi\big(x,\xi',\xi_d = \pi_{j,2}(\varrho'),\tau\big) \neq 0,
$$

by (6.8). From the homogeneity of $b_\varphi^{k-1}(x,\xi',\tau)$ and $\widehat{b}_\varphi^k(x,\xi',\tau)$ in ϱ', it follows that there exists some $C > 0$ such that

$$
\Sigma_{\mathcal{T}}(\varrho', \boldsymbol{z}, \boldsymbol{z}) \geqslant C\big(\lambda_{\mathsf{T},\tau}^3 |z_0|^2 + \lambda_{\mathsf{T},\tau}|z_1|^2\big),
$$

if $\varrho' \in \mathscr{U}_1$. The result of Lemma 6.10 thus follows from Proposition 2.6, having in mind what is exposed in Section 2.5 since we have $\tau \geqslant \kappa_0\sigma$ here. $\qquad\square$

Proof of Lemma 6.11. — We let $u = \mathrm{Op}_\mathsf{T}(\chi)w$. Performing an integration by parts, one has

$$2\,\mathrm{Re}\left(L_1 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+ = 2\,\mathrm{Re}\left(\left(D_d - \mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1})\right)u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+$$
$$= \mathrm{Re}\left(i\left(\Lambda_{\mathsf{T},\tau}^{2s+1}\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1}) - \mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1})^*\Lambda_{\mathsf{T},\tau}^1\right)u, u\right)_+$$
$$+ \mathrm{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1}u_{|x_d=0^+}, u_{|x_d=0^+})_\partial.$$

Note that $\mathrm{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1}u_{|x_d=0^+}, u_{|x_d=0^+})_\partial = |u_{|x_d=0^+}|^2_{s+1/2,\tau}$.

Next, the operator $i\left(\Lambda_{\mathsf{T},\tau}^{2s+1}\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1}) - \mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1})^*\Lambda_{\mathsf{T},\tau}^{2s+1}\right)$ has the following *real* principal symbol

$$\vartheta(\varrho') = -2\,\mathrm{Im}\,\pi_{j,1}(\varrho')\lambda_{\mathsf{T},\tau}^{2s+1},$$

and since $\mathrm{Im}\,\pi_{j,1}(\varrho') \leqslant -C_1\lambda_{\mathsf{T},\tau} < 0$ in $\mathscr{U}$ one obtains $\vartheta(\varrho') \gtrsim \lambda_{\mathsf{T},\tau}^{2s+2}$ in $\mathscr{U}$. Since $\mathscr{U}$ is neighborhood of $\mathrm{supp}(\chi)$, the microlocal Gårding inequality of [31, Th. 2.49] (the proof adapts to the case with parameter σ as explained in Section 2.5 since $\sigma \lesssim \tau$) yields

$$2\,\mathrm{Re}\left(L_1 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+ \geqslant |u_{|x_d=0^+}|^2_{s+1/2,\tau} + C\|\Lambda_{\mathsf{T},\tau}^{s+1}u\|^2_+ - C_N\|w\|^2_{0,-N,\tau},$$

for $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large. With the Young inequality one obtains

$$\left|\left(L_1 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+\right| \lesssim \frac{1}{\varepsilon}\|\Lambda_{\mathsf{T},\tau}^s L_1 u\|^2_+ + \varepsilon\|\Lambda_{\mathsf{T},\tau}^{s+1}u\|^2_+,$$

which yields for ε chosen sufficiently small,

(6.9) $|u_{|x_d=0^+}|_{s+1/2,\tau} + \|u\|_{0,s+1,\tau} \lesssim \|L_1 u\|_{0,s,\tau} + \|w\|_{0,-N,\tau}.$

Finally, we write

(6.10) $\|D_d u\|_{0,s,\tau} \leqslant \|L_1 u\|_{0,s,\tau} + \|\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,1})u\|_{0,s,\tau} \lesssim \|L_1 u\|_{0,s,\tau} + \|u\|_{0,s+1,\tau}.$

Putting together (6.9) and (6.10), the result of Lemma 6.11 follows.   □

Proof of Lemma 6.12. — We let $u = \mathrm{Op}_\mathsf{T}(\chi)w$. Performing an integration by parts, one has

$$2\,\mathrm{Re}\left(L_2 u, -i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+ = 2\,\mathrm{Re}\left(\left(D_d - \mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,2})\right)u, -i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+$$
$$= \mathrm{Re}\left(i\left(\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,2})^*\Lambda_{\mathsf{T},\tau}^1 - \Lambda_{\mathsf{T},\tau}^{2s+1}\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,2})\right)u, u\right)_+$$
$$- \mathrm{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1}u_{|x_d=0^+}, u_{|x_d=0^+})_\partial.$$

Note that $\mathrm{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1}u_{|x_d=0^+}, u_{|x_d=0^+})_\partial = |u_{|x_d=0^+}|^2_{s+1/2,\tau}$.

Next, the operator $i\left(\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,2})^*\Lambda_{\mathsf{T},\tau}^{2s+1} - \Lambda_{\mathsf{T},\tau}^{2s+1}\mathrm{Op}_\mathsf{T}(\widetilde{\chi}\pi_{j,2})\right)$ has the following *real* principal symbol

$$\vartheta(\varrho') = 2\,\mathrm{Im}\,\pi_{j,2}(\varrho')\lambda_{\mathsf{T},\tau}^{2s+1},$$

and since $\mathrm{Im}\,\pi_{j,2}(\varrho') \geqslant C_2\lambda_{\mathsf{T},\tau} > 0$ in $\mathscr{U}$ one obtains $\vartheta(\varrho') \gtrsim \lambda_{\mathsf{T},\tau}^{2s+2}$ in $\mathscr{U}$. Since $\mathscr{U}$ is neighborhood of $\mathrm{supp}(\chi)$, the microlocal Gårding inequality of [31, Th. 2.49] (the proof adapts to the case with parameter σ as explained in Section 2.5 since $\sigma \lesssim \tau$) yields

$$2\,\mathrm{Re}\left(L_2 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\right)_+ \geqslant -|u_{|x_d=0^+}|^2_{s+1/2,\tau} + C\|\Lambda_{\mathsf{T},\tau}^{s+1}u\|^2_+ - C_N\|w\|^2_{0,-N,\tau},$$

for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large. The end of the proof is then similar to that of Lemma 6.11. $\qquad\square$

6.3.2. *Case (ii): one real root.* — One has $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) = 0$ and $\operatorname{Im} \pi_{j,1}(\varrho^{0\prime}) < 0$.

Since the Lopatinskiĭ-Šapiro condition holds for (Q_σ^j, B, φ) at $\varrho^{0\prime}$, by Lemma 6.8 one has

$$b_\varphi(x^0, \xi^{0\prime}, \xi_d = \pi_{j,2}(\varrho^{0\prime}), \tau^0) = b(x^0, \xi^{0\prime} + i\tau^0 d_{x'}\varphi(x^0), i\alpha_j(\varrho^{0\prime})) \neq 0.$$

As the roots $\pi_{j,1}$ and $\pi_{j,2}$ are locally smooth with respect to $\varrho' = (x, \xi', \tau, \sigma)$ and homogeneous of degree one in (ξ', τ, σ), there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_1 > 0$, $C_2 > 0$ such that

$$\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}} \mid |\xi'|^2 + \tau^2 + \sigma^2 = 1\}$$

is compact and

$$\tau \geqslant \kappa_0 \sigma, \quad \pi_{j,1}(\varrho') \neq \pi_{j,2}(\varrho'),$$
$$\operatorname{Im} \pi_{j,2}(\varrho') \geqslant -C_2 \lambda_{\tau,\tau}, \quad \text{and} \quad \operatorname{Im} \pi_{j,1}(\varrho') \leqslant -C_1 \lambda_{\tau,\tau},$$

and

(6.11) $$b_\varphi(x, \xi', \xi_d = \pi_{j,2}(\varrho'), \tau) \neq 0,$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

We let $\chi \in S^0_{\tau,\tau}$ and $\widetilde{\chi} \in S^0_{\tau,\tau}$ be homogeneous of degree zero in the variable (ξ', τ, σ) and be such that $\operatorname{supp}(\widetilde{\chi}) \subset \mathscr{U}$ and $\widetilde{\chi} \equiv 1$ on $\operatorname{supp}(\chi)$. From the smoothness and the homogeneity of the roots, one has $\widetilde{\chi}\pi_{j,k} \in S^1_{\tau,\tau}$, $k = 1, 2$. We set

$$L_2 = D_d - \operatorname{Op}_\tau(\widetilde{\chi}\pi_{j,2}) \quad \text{and} \quad L_1 = D_d - \operatorname{Op}_\tau(\widetilde{\chi}\pi_{j,1}).$$

Lemma 6.10 and Lemma 6.11 also apply in Case (ii) and we shall use them. In addition to these two lemmas we shall need the following lemma.

LEMMA 6.13. — *There exist $C > 0$, $\tau_0 > 0$ such that*

$$\tau^{-1/2}\|w\|_{2,\tau} \leqslant C\big(\|Q^j_{\sigma,\varphi}w\|_+ + |\operatorname{tr}(w)|_{1,1/2,\tau}\big),$$

for $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$ and $w \in \overline{\mathscr{C}}^\infty_c(W_+)$.

Proving Lemma 6.13 is fairly classical, based on writing $Q^j_{\sigma,\varphi} = Q^j_s + iQ_a$ and on an expansion of $\|Q^j_{\sigma,\varphi}w\|^2_+$ and some integration by parts. We provide the details in the proof below as the occurrence of the parameter σ is not that classical. Lemma 6.13 expresses the loss of a half-derivative if one root, here $\pi_{j,2}$, is real.

Observing that

$$L_1 \operatorname{Op}_\tau(\chi) L_2 = \operatorname{Op}_\tau(\chi) L_1 L_2 \mod \Psi^{1,0}_\tau$$
$$= \operatorname{Op}_\tau(\chi) Q^j_{\sigma,\varphi} \mod \Psi^{1,0}_\tau,$$

and applying Lemma 6.11 to $w = L_2 v$, one obtains

$$|\mathrm{Op}_\tau(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} \lesssim \|L_1 \mathrm{Op}_\tau(\chi) L_2 v\|_+ + \|v\|_{1,-N,\tau}$$
$$\lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{1,\tau},$$

for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large. We set $u = \mathrm{Op}_\tau(\chi) v$, and using the trace inequality

$$|w_{|x_d=0^+}|_{s,\tau} \lesssim \|w\|_{s+1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } s > 0,$$

we have

$$|L_2 u_{|x_d=0^+}|_{1/2,\tau} \lesssim |\mathrm{Op}_\tau(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + |v_{|x_d=0^+}|_{1/2,\tau}$$
$$\lesssim |\mathrm{Op}_\tau(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + \|v\|_{1,\tau}.$$

Therefore, we obtain

$$|L_2 u_{|x_d=0^+}|_{1/2,\tau} \lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{1,\tau}.$$

On the one hand, together with Lemma 6.10, one has the estimate

(6.12) $\qquad \big|\mathrm{tr}(u)\big|_{1,1/2,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{2,-1,\tau},$

for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large using the following trace inequality

$$|\mathrm{tr}(w)|_{m,s,\tau} \lesssim \|w\|_{m+1,s-1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } m \in \mathbb{N}, \; s \in \mathbb{R}.$$

On the other hand, with Lemma 6.13 one has

$$\tau^{-1/2}\|u\|_{2,\tau} \lesssim \|Q_{\sigma,\varphi}^j u\|_+ + |\mathrm{tr}(u)|_{1,1/2,\tau},$$

again for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large and since $[Q_{\sigma,\varphi}^j, \mathrm{Op}_\tau(\chi)] \in \Psi_\tau^{1,0}$ one finds

(6.13) $\qquad \tau^{-1/2}\|u\|_{2,\tau} \lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{1,\tau} + |\mathrm{tr}(u)|_{1,1/2,\tau},$

Now, with $\varepsilon > 0$ chosen sufficiently small one computes (6.12)$+\varepsilon \times$(6.13) and obtains

$$\tau^{-1/2}\|u\|_{2,\tau} + |\mathrm{tr}(u)|_{1,1/2,\tau} \lesssim \|Q_{\sigma,\varphi}^j v\|_+ + |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}.$$

As $u = \mathrm{Op}_\tau(\chi) v$, with a commutator argument we obtain

$$|B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} \lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + |\mathrm{tr}(v)|_{1,-1/2,\tau}$$
$$\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}.$$

yielding (6.6) and thus concluding the proof of Proposition 6.9 in Case (ii). $\qquad \square$

We now provide a proof of Lemma 6.13.

Proof of Lemma 6.13. — We recall that $Q_{\sigma,\varphi}^j = Q_s^j + i Q_a$, yielding

(6.14) $\qquad \|Q_{\sigma,\varphi}^j w\|_+^2 = \|Q_s^j w\|_+^2 + \|Q_a w\|_+^2 + 2\,\mathrm{Re}(Q_s^j w, i Q_a w)_+.$

With the integration by parts formula

$$(f, D_d g)_+ = (D_d f, g)_+ - i(f_{|x_d=0^+}, g_{|x_d=0^+})_\partial,$$

and the forms of Q_s^j and Q_a given in (6.2) one has

$$(f, Q_s^j g)_+ = (Q_s^j f, g)_+ - i(f_{|x_d=0^+}, D_d g_{|x_d=0^+})_\partial - i(D_d f_{|x_d=0^+}, g_{|x_d=0^+})_\partial,$$

and
$$(f, Q_a g)_+ = (Q_a f, g)_+ - 2\tau i(\partial_d \varphi f_{|x_d=0^+}, g_{|x_d=0^+})_\partial,$$

yielding
$$(Q_a w, Q_s^j w)_+ = (Q_s^j Q_a w, w)_+ - i(Q_a w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial$$
$$- i(D_d Q_a w_{|x_d=0^+}, w_{|x_d=0^+})_\partial,$$
$$(Q_s^j w, Q_a w)_+ = (Q_a Q_s^j w, w)_+ - 2i\tau(\partial_d \varphi Q_s^j w_{|x_d=0^+}, w_{|x_d=0^+})_\partial.$$

This gives

(6.15)
$$2\operatorname{Re}(Q_s^j w, i Q_a w)_+ = i([Q_s^j, Q_a] w, w)_+ + \tau \mathcal{A}(w),$$

with

(6.16)
$$\mathcal{A}(w) = \tau^{-1}(Q_a w, D_d w)_\partial + \tau^{-1}\big((D_d Q_a - 2\tau \partial_d \varphi Q_s^j) w, w\big)_\partial.$$

We have the following lemma adapted from Lemma 3.25 in [31].

Lemma 6.14. — *The operators $Q_a \in \tau \mathscr{D}^1$ and $D_d Q_a - 2\tau \partial_d \varphi Q_s^j \in \mathscr{D}_\tau^3$ can be cast in the following forms*

$$Q_a = 2\tau \partial_d \varphi D_d + 2\widetilde{r}(x, D', \tau d_{x'}\varphi) \mod \tau \mathscr{D}^0,$$

and

$$D_d Q_a - 2\tau \partial_d \varphi Q_s^j = -2\tau \partial_d \varphi\big(R(x, D') + (-1)^j \sigma^2 - (\tau \partial_d \varphi)^2 - r(x, \tau d_{x'}\varphi)\big)$$
$$+ 2\widetilde{r}(x, D', \tau d_{x'}\varphi) D_d \mod \tau \Psi_\tau^{1,0}.$$

With this lemma we find
$$\mathcal{A}(w) = 2(\partial_d \varphi D_d w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial$$
$$+ 2(\widetilde{r}(x, D', d_{x'}\varphi) w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial$$
$$+ 2\big(\widetilde{r}(x, D', d_{x'}\varphi) D_d w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial$$
(6.17)
$$- 2\big(\partial_d \varphi\big(R(x, D') + (-1)^j \sigma^2\big) w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial$$
$$+ 2\big(\partial_d \varphi\big((\tau \partial_d \varphi)^2 + r(x, \tau d_{x'}\varphi)\big) w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial$$
$$+ (\operatorname{Op}(c_0) w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial$$
$$+ \big((\operatorname{Op}(\widetilde{c}_0) D_d + \operatorname{Op}(c_1)) w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial,$$

with $\operatorname{Op}(c_0), \operatorname{Op}(\widetilde{c}_0) \in \mathscr{D}^0$ and $\operatorname{Op}(c_1) \in \mathscr{D}_{\mathsf{T},\tau}^1$. Observe that one has

(6.18)
$$|\mathcal{A}(w)| \lesssim |\operatorname{tr}(w)|_{1,0,\tau}^2.$$

From (6.14) and (6.15) one writes

(6.19)
$$\|Q_{\sigma,\varphi}^j w\|_+^2 + \tau |\operatorname{tr}(w)|_{1,0,\tau}^2 \gtrsim \|Q_s^j w\|_+^2 + \|Q_a w\|_+^2 + \operatorname{Re}(i[Q_s^j, Q_a] w, w)_+.$$

We now use the following lemma whose proof is given below.

Lemma 6.15. — *There exists $C, C' > 0$, $\mu > 0$ and $\tau_0 > 0$ such that*

$$\mu\big(\|Q_s^j w\|_+^2 + \|Q_a w\|_+^2\big) + \tau \operatorname{Re}(i[Q_s^j, Q_a]w, w)_+ \geqslant C\|w\|_{2,\tau}^2 - C'|\operatorname{tr}(w)|_{1,1/2,\tau}^2,$$

for $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$ and $w \in \overline{\mathscr{C}}_c^\infty(W_+)$

Let $\mu > 0$ be as in Lemma 6.15 and let $\tau > 0$ be such that $\mu \tau^{-1} \leqslant 1$. From (6.19) one then writes

$$\|Q_{\sigma,\varphi}^j w\|_+^2 + \tau |\operatorname{tr}(w)|_{1,0,\tau}^2 \gtrsim \tau^{-1}\Big(\mu\big(\|Q_s^j w\|_+^2 + \|Q_a w\|_+^2\big) + i\tau([Q_s^j, Q_a]w, w)_+\Big),$$

which with Lemma 6.15 yields the result of Lemma 6.13 using that $\tau|\operatorname{tr}(w)|_{1,0,\tau} \lesssim |\operatorname{tr}(w)|_{1,1/2,\tau}$. □

Proof of Lemma 6.15. — One has $[Q_s^j, Q_a] \in \tau \mathscr{D}_\tau^2$. Writing

$$\tau \operatorname{Re}\big(i[Q_s^j, Q_a]w, w\big)_+ = \operatorname{Re}\big(i\tau^{-1}[Q_s^j, Q_a]w, \tau^2 w\big)_+,$$

it can be seen as a interior differential quadratic form of type $(2, 0)$ as in Definition 2.1. Therefore

$$T(w) = \mu\big(\|Q_s^j w\|_+^2 + \|Q_a w\|_+^2\big) + \tau \operatorname{Re}\big(i[Q_s^j, Q_a]w, w\big)_+$$

is also an interior differential quadratic form of this type with principal symbol given by

$$t(\varrho) = \mu|q_{\sigma,\varphi}^j(\varrho)|^2 + \tau\{q_s^j, q_a\}(\varrho), \qquad \varrho = (x, \xi, \tau, \sigma).$$

Let $\tau_0 > 0$. By Lemma 6.4, the sub-ellipticity property of (Q_σ^j, φ) implies

$$t(\varrho) \gtrsim \lambda_\tau^4, \qquad \varrho \in \overline{W} \times \mathbb{R}^d \times [0, +\infty) \times [0, +\infty), \ \tau \geqslant \tau_0 \sigma,$$

for $\mu > 0$ chosen sufficiently large. The Gårding inequality of proposition 2.3 yields

$$T(w) \geqslant C\|w\|_{2,\tau}^2 - C'|\operatorname{tr}(w)|_{1,1/2,\tau}^2,$$

for some $C, C' > 0$ and for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large. □

6.3.3. *Case (iii): both roots lying in the lower complex half-plane*. — The result in the present case is a simple consequence of the general result given in Lemma A.1 whose proof can be found in [8]. In the second order case however, the proof does not require the same level of technicality.

One has $\operatorname{Im} \pi_{j,1}(\varrho^{0\prime}) < 0$ and $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) < 0$. As the roots $\pi_{j,1}$ and $\pi_{j,2}$ depend continuously on the variable $\varrho' = (x, \xi', \tau, \sigma)$, there exists a conic open neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_0 > 0$ such that

$$\tau \geqslant \kappa_0 \sigma, \quad \operatorname{Im} \pi_{j,1}(\varrho') \leqslant -C_0 \lambda_{\mathrm{T},\tau}, \quad \text{and} \ \operatorname{Im} \pi_{j,2}(\varrho') \leqslant -C_0 \lambda_{\mathrm{T},\tau},$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

Let $\chi \in S_{\mathrm{T},\tau}^0$ be as in the statement of Proposition 6.9 and set $u = \operatorname{Op}_{\mathrm{T}}(\chi)v$.

We recall that $Q_{\sigma,\varphi}^j = Q_s^j + iQ_a$, yielding

$$\|Q_{\sigma,\varphi}^j u\|_+^2 = \|Q_s^j u\|_+^2 + \|Q_a u\|_+^2 + 2\operatorname{Re}(Q_s^j u, iQ_a u)_+.$$

We set $L(u) = \|Q_s^j u\|_+^2 + \|Q_a u\|_+^2$. This is an interior differential quadratic form in the sense of Definition 2.1. Its principal symbol is given by

$$\ell(\varrho) = (q_s^j)(\varrho)^2 + q_a(\varrho)^2, \qquad \varrho = (x, \xi, \tau, \sigma).$$

For $\varepsilon \in (0, 1)$ we write

$$(6.20) \qquad \|Q_{\sigma, \varphi}^j u\|_+^2 \geqslant \varepsilon L(u) + 2 \operatorname{Re}(Q_s^j u, i Q_a u)_+.$$

For concision we write $\varrho = (\varrho', \xi_d)$ with $\varrho' = (x, \xi', \tau, \sigma)$. The set

$$\mathscr{L} = \{\varrho = (\varrho', \xi_d) \mid \varrho' \in \overline{\mathscr{U}}, \ \xi_d \in \mathbb{R}, \text{ and } |\xi|^2 + \tau^2 + \sigma^2 = 1\}$$

is compact recalling that W is bounded. On $\mathscr{L}$ one has $|q_{\sigma, \varphi}^j(\varrho)| \geqslant C > 0$. By homogeneity one has

$$(6.21) \qquad |q_{\sigma, \varphi}^j(\varrho)| \gtrsim \lambda_\tau^2, \qquad \varrho' \in \mathscr{U}, \ \xi_d \in \mathbb{R}, \ \text{ if } \tau \geqslant \tau_0 \sigma,$$

for some $\tau_0 > 0$. Therefore

$$(6.22) \qquad \ell(\varrho) \gtrsim \lambda_\tau^4, \qquad \varrho' \in \mathscr{U}, \ \xi_d \in \mathbb{R}, \ \text{ if } \tau \geqslant \tau_0 \sigma.$$

By the Gårding inequality of Proposition 2.2 one obtains

$$(6.23) \qquad \operatorname{Re} L(u) \geqslant C \|u\|_{2,\tau}^2 - C' |\operatorname{tr}(u)|_{1,1/2}^2 - C_N \|v\|_{2,-N,\tau}^2,$$

for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large.

From the proof of Lemma 6.13 one has

$$(6.24) \qquad 2 \operatorname{Re}(Q_s^j u, i Q_a u)_+ = i([Q_s^j, Q_a]u, u)_+ + \tau \mathcal{A}(u)$$

with the boundary quadratic form $\mathcal{A}$ given in (6.16)–(6.17).

On the one hand, one has $[Q_s^j, Q_a] \in \tau \mathscr{D}_\tau^2$ and therefore

$$(6.25) \qquad |\operatorname{Re}([Q_s^j, Q_a]u, u)_+| \lesssim \tau \|u\|_{2,-1,\tau}^2 \lesssim \tau^{-1} \|u\|_{2,\tau}^2.$$

On the other hand, we have the following lemma that provides a microlocal positivity property for the boundary quadratic form $\mathcal{A}$. A proof is given below.

LEMMA 6.16. — *There exist C, C_N and $\tau_0 > 0$ such that*

$$\tau \operatorname{Re} \mathcal{A}(u) \geqslant C |\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C_N |\operatorname{tr}(v)|_{1,-N,\tau}^2, \quad \text{ for } u = \operatorname{Op}_\tau(\chi)v,$$

for $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$.

With (6.24)–(6.25), and Lemma 6.16 one obtains

$$2 \operatorname{Re}(Q_s^j u, i Q_a u)_+ \geqslant C |\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C' \tau^{-1} \|u\|_{2,\tau}^2 - C_N |\operatorname{tr}(v)|_{1,-N,\tau}^2$$

$$(6.26) \qquad \geqslant C |\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C' \tau^{-1} \|u\|_{2,\tau}^2 - C_N' \|v\|_{2,-N,\tau}^2,$$

with a trace inequality, for $\tau \geqslant \kappa_0 \sigma$ chosen sufficiently large.

With (6.20), (6.23), and (6.26) one obtains

$$\|Q_{\sigma, \varphi}^j u\|_+^2 \geqslant \varepsilon C \|u\|_{2,\tau}^2 - C' \varepsilon |\operatorname{tr}(u)|_{1,1/2}^2 - C_N \varepsilon \|v\|_{2,-N,\tau}^2$$

$$+ C |\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C' \tau^{-1} \|u\|_{2,\tau}^2 - C_N' \|v\|_{2,-N,\tau}^2.$$

With ε chosen sufficiently small and $\tau \geqslant \kappa_0 \sigma$ sufficiently large one obtains for any $N \in \mathbb{N}$

$$\|u\|_{2,\tau} + |\operatorname{tr}(u)|_{1,1/2,\tau} \lesssim \|Q^j_{\sigma,\varphi} u\|_+ + \|v\|_{2,-N,\tau}.$$

With a commutator argument, as $u = \operatorname{Op}_\tau(\chi)v$ one finds $\|Q^j_{\sigma,\varphi}u\|_+ \lesssim \|Q^j_{\sigma,\varphi}v\|_+ + \|v\|_{2,-1,\tau}$, yielding estimate (6.7) and thus concluding the proof of Proposition 6.9 in Case (iii). $\qquad\square$

Proof of Lemma 6.16. — With (6.17) one sees that it suffices to consider the following boundary quadratic form

$$\begin{aligned}
\widetilde{\mathcal{A}}(w) = {} & 2(\partial_d\varphi D_d w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial \\
& + 2(\widetilde{r}(x,D',d_{x'}\varphi)w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial \\
& + 2\big(\widetilde{r}(x,D',d_{x'}\varphi)D_d w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial \\
& - 2\big(\partial_d\varphi\big(R(x,D') + (-1)^j\sigma^2\big)w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial, \\
& + 2\big(\partial_d\varphi\big((\tau\partial_d\varphi)^2 + r(x,\tau d_{x'}\varphi)\big)w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial,
\end{aligned}$$

in place of $\mathcal{A}$. It is of type $(1,0)$ in the sense of Definition 2.4. Its principal symbol is given by $a_0(\varrho',\xi_d,\xi_d') = (1,\xi_d)A(\varrho')^t(1,\xi_d)$ with

$$A(\varrho') = 2\begin{pmatrix} A_{11}(\varrho') & \widetilde{r}(x,\xi',d_{x'}\varphi)_{|x_d=0^+} \\ \widetilde{r}(x,\xi',d_{x'}\varphi)_{|x_d=0^+} & \partial_d\varphi_{|x_d=0^+} \end{pmatrix},$$

where

$$A_{11}(\varrho') = -(\partial_d\varphi)\big(r(x,\xi') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 - r(x,\tau d_{x'}\varphi)\big)_{|x_d=0^+}$$

with $\varrho' = (x,\xi',\tau,\sigma)$. The associated bilinear symbol introduced in (2.4) is given by

$$\Sigma_{\mathcal{A}}(\varrho',\boldsymbol{z},\boldsymbol{z}') = \boldsymbol{z}A(\varrho')^t\overline{\boldsymbol{z}}', \qquad \boldsymbol{z} = (z_0,z_1) \in \mathbb{C}^2, \ \boldsymbol{z}' = (z_0',z_1') \in \mathbb{C}^2.$$

One computes

$$\begin{aligned}
\det A(\varrho') = -4\Big(&(\partial_d\varphi)^2\big(r(x,\xi') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 \\
& - r(x,\tau d_{x'}\varphi)\big) + \widetilde{r}(x,\xi',d_{x'}\varphi)^2\Big)_{|x_d=0^+}.
\end{aligned}$$

With Lemma 4.3 one sees that $\operatorname{Im}\pi_{j,2} < 0$ is equivalent to having $\det A(\varrho') > 0$. We thus have

$$\det A(\varrho') \geqslant C > 0, \quad \text{for } \varrho' = (x,\xi',\tau,\sigma) \in \mathbb{S}_{\overline{\mathscr{U}}},$$

with $\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}} \mid \xi_d \in \mathbb{R}, |\xi|^2 + \tau^2 + \sigma^2 = 1\}$ since $\mathbb{S}_{\overline{\mathscr{U}}}$ is compact. Since $\partial_d\varphi_{|x_d=0^+} \geqslant C' > 0$ then one finds that

$$\operatorname{Re}\Sigma_{\mathcal{A}}(\varrho',\boldsymbol{z},\boldsymbol{z}) \geqslant C(|z_0|^2 + |z_1|^2), \quad \varrho' = (x,\xi',\tau,\sigma) \in \overline{\mathscr{U}}, \ |(\xi',\tau,\sigma)| = 1.$$

By homogeneity one obtains

$$\operatorname{Re}\Sigma_{\mathcal{A}}(\varrho',\boldsymbol{z},\boldsymbol{z}) \geqslant C(\lambda^2_{\tau,\tau}|z_0|^2 + |z_1|^2), \quad \varrho' = (x,\xi',\tau,\sigma) \in \overline{\mathscr{U}}, \ |(\xi',\tau,\sigma)| \geqslant 1.$$

With Proposition 2.6, having in mind what is exposed in Section 2.5 since we have $\tau \geqslant \kappa_0\sigma$ here, one obtains

$$\operatorname{Re}\widetilde{\mathcal{A}}(u) \geqslant C|\operatorname{tr}(u)|_{1,0,\tau}^2 - C_N|\operatorname{tr}(v)|_{1,-N,\tau}^2, \quad \text{for } u = \operatorname{Op}_\tau(\chi)v,$$

for $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large.

Here, we have $\operatorname{Im}\pi_{j,2} < 0$ and thus $|\xi'| \lesssim \tau$ by Lemma 4.6. Thus one has

$$\tau|\operatorname{tr}(u)|_{1,0,\tau}^2 \gtrsim |\operatorname{tr}(u)|_{1,1/2,\tau}^2 - |\operatorname{tr}(v)|_{1,-N,\tau}^2,$$

by the microlocal Gårding inequality, for instance invoking Proposition 2.6 for a boundary quadratic form of type $(1, 1/2)$. This concludes the proof. $\qquad\square$

7. Local Carleman estimate for the fourth-order operator

In Proposition 5.1, we proved that a norm of all traces at the boundary could be estimates from the values taken by the boundary operators. That result relied on the Lopatinskiĭ-Šapiro condition. Here we aim to moreover estimate a volume norm. The strategy is to estimate this norm from all traces at the boundary and not from the boundary operators. The proof is then much simpler. Yet combined with Proposition 5.1 an estimation of both volume and trace norms will be obtained from the the values taken by the boundary operators.

7.1. A first estimate

PROPOSITION 7.1. — *Let $\kappa_0' > \kappa_1' > \kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d\varphi \geqslant C > 0$ in W and such that (Q_σ^j, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2. Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ with $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ nonvanishing with $\kappa_1\sigma^0 \leqslant \tau^0 \leqslant \kappa_1'\sigma^0$.*

Then, there exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$ such that if $\chi \in S_{\tau,\tau}^0$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$(7.1) \qquad \tau^{-1/2}\|\operatorname{Op}_\tau(\chi)v\|_{4,\tau} \leqslant C\big(\|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}\big),$$

for $\tau \geqslant \tau_0$, $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$, and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

The proof of Proposition 7.1 is based on the microlocal results of Proposition 6.9.

REMARK 7.2. — An important aspect is that here we have $\sigma \gtrsim \tau$; it is key to have only a loss of a half-derivative. Losses are due to lack of ellipticity, that is, having root(s) of $p_{\sigma,\varphi}$ lying on the real axis. If $\sigma = 0$ the operator is a square and two roots can lie and the real axis yielding a loss of a full derivative.

Note that the issue of having two potential roots on the real axis and thus a loss of a full derivative is independent of the choice of boundary operators.

Having $\sigma > 0$ implies that only one root of $p_{\sigma,\varphi}$ can lie on the real axis. In the proof we shall write P_σ as a factor of two operators and we apply the Carleman

estimates of Proposition 7.1 to each one. Each estimates exhibits a loss of a half-derivative. If naively concatenated the two estimates indeed yield a Carleman estimate with a loss of a full derivative. In fact the loss of each estimate occurs in different microlocal regions and a microlocal concatenation allows one to have a loss of only a half derivative.

Having $\sigma > 0$ was not sufficient for us to separate microlocal regions if τ increases because of homogeneity issues. This explains the introduction of the condition is $\sigma \gtrsim \tau$. We do not exclude that $\sigma > 0$ could suffice to reach a similar estimate; we could not prove it.

Proof. — We shall concatenate the estimates of Proposition 6.9 for $Q^1_{\sigma,\varphi}$ and $Q^2_{\sigma,\varphi}$ with the boundary operator B simply given by the Dirichlet trace operator,

$$Bu_{|x_d=0^+} = u_{|x_d=0^+}.$$

One has $b(x,\xi) = 1$ and $b_\varphi(x,\xi',\xi_d,\tau) = 1$. Since $\partial_d\varphi > 0$ then $\operatorname{Im}\pi_{j,1} < 0$. Thus, either $q^{j,+}_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = 1$ or $q^{j,+}_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = \xi_d - \pi_{j,2}$. With Lemma 6.8 one sees that the Lopatinskiĭ-Šapiro holds for $(Q^1_{\sigma,\varphi}, B, \varphi)$ and $(Q^2_{\sigma,\varphi}, B, \varphi)$ at $\varrho^{0\prime}$.

Proposition 6.9 thus applies. Let $\mathscr{U}_j$ be the conic neighborhood of $\varrho^{0\prime}$ obtained invoking this proposition for $Q^j_{\sigma,\varphi}$, for $j = 1$ or 2. In $\mathscr{U}_j$ one has $\tau \geqslant \kappa_0\sigma$. We set

$$\mathscr{U} = \mathscr{U}_1 \cap \mathscr{U}_2 \cap \{\tau \leqslant \kappa'_0\sigma\},$$

and we consider $\chi \in S^0_{\tau,\tau}$, homogeneous of degree 0 in (ξ',τ,σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$.

Since in $\mathscr{U}$ one has $\sigma > 0$ then $\pi_{1,2}$ and $\pi_{2,2}$ cannot be both real by Lemma 4.7. Proposition 6.9 thus implies that we necessarily have the following two estimates

$$(7.2) \qquad \tau^{-\ell_1}\|\mathrm{Op}_\tau(\chi)w\|_{2,\tau} \lesssim \|Q^1_{\sigma,\varphi}w\|_+ + |w_{|x_d=0^+}|_{3/2,\tau} + \|w\|_{2,-1,\tau},$$

and

$$(7.3) \qquad \tau^{-\ell_2}\|\mathrm{Op}_\tau(\chi)w\|_{2,\tau} \lesssim \|Q^2_{\sigma,\varphi}w\|_+ + |w|_{3/2,\tau} + \|w\|_{2,-1,\tau},$$

with either $(\ell_1,\ell_2) = (1/2,0)$ or $(\ell_1,\ell_2) = (0,1/2)$, for $w \in \overline{\mathscr{C}}^\infty_c(W_+)$ and $\tau \geqslant \kappa_0\sigma$ chosen sufficiently large.

Let us assume that $(\ell_1,\ell_2) = (1/2,0)$. The other case can be treated similarly. Writing $P_{\sigma,\varphi} = Q^2_{\sigma,\varphi}Q^1_{\sigma,\varphi}$, with (7.3) one has

$$\|\mathrm{Op}_\tau(\chi)Q^1_{\sigma,\varphi}v\|_{2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |Q^1_{\sigma,\varphi}v_{|x_d=0^+}|_{3/2,\tau} + \|v\|_{4,-1,\tau}$$
$$\lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{2,3/2,\tau} + \|v\|_{4,-1,\tau}.$$

Since $[\mathrm{Op}_\tau(\chi), Q^1_{\sigma,\varphi}] \in \Psi^{1,0}_\tau$ one finds

$$(7.4) \qquad \|Q^1_{\sigma,\varphi}\mathrm{Op}_\tau(\chi)v\|_{2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}.$$

For $k = 0, 1$ or 2, one writes

$$\|Q^1_{\sigma,\varphi}\mathrm{Op}_\tau(\chi)D^k_d\Lambda^{2-k}_{\tau,\tau}v\|_+ + |\operatorname{tr}(\mathrm{Op}_\tau(\chi)D^k_d\Lambda^{2-k}_{\tau,\tau}v)|_{1,1/2,\tau} + \|\mathrm{Op}_\tau(\chi)D^k_d\Lambda^{2-k}_{\tau,\tau}v\|_{2,-1,\tau}$$
$$\lesssim \|Q^1_{\sigma,\varphi}\mathrm{Op}_\tau(\chi)v\|_{2,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau},$$

since $[Q^1_{\sigma,\varphi}\mathrm{Op}_\tau(\chi), D^k_d\Lambda^{2-k}_{\tau,\tau}] \in \Psi^{4,-1}_\tau$.

Let $\widetilde{\chi} \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree zero in the variable (ξ', τ, σ) and be such that $\mathrm{supp}(\widetilde{\chi}) \subset \mathscr{U}$ and $\widetilde{\chi} \equiv 1$ on a neighborhood of $\mathrm{supp}(\chi)$. With (7.2), from (7.4) one thus obtains

$$\tau^{-1/2} \| \mathrm{Op}_{\mathsf{T}}(\widetilde{\chi}) \mathrm{Op}_{\mathsf{T}}(\chi) D_d^k \Lambda_{\mathsf{T},\tau}^{2-k} v \|_{2,\tau} \lesssim \| P_{\sigma,\varphi} v \|_+ + | \mathrm{tr}(v) |_{3,1/2,\tau} + \| v \|_{4,-1,\tau}.$$

Since $\mathrm{Op}_{\mathsf{T}}(\widetilde{\chi}) \mathrm{Op}_{\mathsf{T}}(\chi) D_d^k \Lambda_{\mathsf{T},\tau}^{2-k} = \Lambda_{\mathsf{T},\tau}^{2-k} D_d^k \mathrm{Op}_{\mathsf{T}}(\chi) \mod \Psi_\tau^{2,-1}$ one deduces

$$\tau^{-1/2} \| D_d^k \mathrm{Op}_{\mathsf{T}}(\chi) v \|_{2,2-k,\tau} \lesssim \| P_{\sigma,\varphi} v \|_+ + | \mathrm{tr}(v) |_{3,1/2,\tau} + \| v \|_{4,-1,\tau}.$$

Using that $k = 0, 1$ or 2, the result follows. $\qquad\square$

Consequence of this microlocal result is the following local result by means of a patching procedure as for the proof of Proposition 5.1 in Section 5.2.

PROPOSITION 7.3. — *Let $\kappa'_0 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geqslant C > 0$ in W and such that (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2.*

Then, there exists a neighborhood W^0 of x^0, $C > 0$, $\tau_0 > 0$ such that

$$(7.5) \qquad \tau^{-1/2} \| v \|_{4,\tau} \leqslant C \big(\| P_{\sigma,\varphi} v \|_+ + | \mathrm{tr}(v) |_{3,1/2,\tau} \big),$$

for $\tau \geqslant \tau_0$, $\kappa_0 \sigma \leqslant \tau \leqslant \kappa'_0 \sigma$, and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

7.2. FINAL ESTIMATE. — Combining the local results of Section 5 for the estimation of the boundary norm under the Lopatinskiĭ-Šapiro condition and the previous local result without any prescribed boundary condition we obtain the Carleman estimate of Theorem 1.2. For a precise statement we write the following theorem.

THEOREM 7.4 (local Carleman estimate for P_σ). — *Let $\kappa'_0 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geqslant C > 0$ in W and such that (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2.*

Assume that $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geqslant \kappa_0 \sigma$.

Then, there exists a neighborhood W^0 of x^0, $C > 0$, $\tau_0 > 0$ such that

$$(7.6) \quad \tau^{-1/2} \| e^{\tau\varphi} u \|_{4,\tau} + | \mathrm{tr}(e^{\tau\varphi} u) |_{3,1/2,\tau}$$
$$\leqslant C \Big(\| e^{\tau\varphi} P_\sigma u \|_+ + \sum_{j=1}^2 | e^{\tau\varphi} B_j v_{|x_d=0+} |_{7/2-k_j,\tau} \Big),$$

for $\tau \geqslant \tau_0$, $\kappa_0 \sigma \leqslant \tau \leqslant \kappa'_0 \sigma$, and $u \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The notation of the function space $\overline{\mathscr{C}}_c^\infty(W_+^0)$ is introduced in (1.6). For the application of this theorem, one has to design a weight function that yields the two important properties: sub-ellipticity and the Lopatinskiĭ-Šapiro condition. Sub-ellipticity is

obtained by means of Proposition 6.3; the Lopatinskiĭ-Šapiro condition by means of Proposition 4.2.

Proof of Theorem 7.4. — The assumption of the theorem allows one to invoke both Propositions 5.1 and 7.3 yielding the existence of a neighborhood W^0 of x^0 where, by Proposition 5.1, one has

$$(7.7) \qquad |\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau},$$

for $\sigma \geqslant 0$, $\tau \geqslant \max(\tau_1,\kappa_0\sigma)$ for some $\tau_1 > 0$ and $v \in \overline{\mathscr{C}}_c^\infty(W_+^0)$. With the Proposition 7.3 one also has

$$(7.8) \qquad\qquad \tau^{-1/2}\|v\|_{4,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau},$$

for $\tau \geqslant \tau_1'$ and $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$ for some $\tau_1' > 0$.

Consider $\sigma > 0$ and $\tau \geqslant \max(\tau_1,\tau_1')$ such that $\kappa_0\sigma \leqslant \tau \leqslant \kappa_0'\sigma$. Combined together (7.7) and (7.8) yield

$$\tau^{-1/2}\|v\|_{4,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau}.$$

Since $\|v\|_{4,-1,\tau} \ll \tau^{-1/2}\|v\|_{4,\tau}$ for τ large one obtains

$$\tau^{-1/2}\|v\|_{4,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau}.$$

If we set $v = e^{\tau\varphi}u$ then the conclusion follows.   □

8. Global Carleman estimate and observability

Using the local Carleman estimate of Theorem 7.4 we prove a global version of this estimate. This allows us to obtain an observability inequality with observation in some open subset $\mathscr{O}$ of Ω. In turn in Section 10 we use this latter inequality to obtain a resolvent estimate for the plate semigroup generator that allows one to deduce a stabilization result for the damped plate equation.

8.1. A global Carleman estimate. — Assume that the Lopatinskiĭ-Šapiro condition of Definition 3.1 holds for (P_0, B_1, B_2) on $\partial\Omega$.

Let $\mathscr{O}_0, \mathscr{O}_1, \mathscr{O}$ be open sets such that $\mathscr{O}_0 \Subset \mathscr{O}_1 \Subset \mathscr{O} \Subset \Omega$. With Proposition 3.31 and Remark 3.32 in [31] there exists $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ such that

(1) $\psi = 0$ and $\partial_\nu\psi < -C_0 < 0$ on $\partial\Omega$;
(2) $\psi > 0$ in Ω;
(3) $d\psi \neq 0$ in $\Omega \setminus \mathscr{O}_0$.

Then, by Proposition 6.3, for γ chosen sufficiently large, one finds that $\varphi = \exp(\gamma\psi)$ is such that a

(1) $\varphi = 1$ and $\partial_\nu\varphi < -C_0 < 0$ on $\partial\Omega$;
(2) $\varphi > 1$ in Ω;
(3) (Q_σ^j,φ) satisfies the sub-ellipticity condition in $\Omega \setminus \mathscr{O}_0$, for $j = 1, 2$, for $\tau \geqslant \tau_0\sigma$ for τ_0 chosen sufficiently large.

Then, with Proposition 4.2, for $\kappa_0 > 0$ chosen sufficiently large one finds that the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at any (x, ξ', τ, σ) for any $x \in \partial\Omega$, $\xi' \in T_x^* \partial\Omega \simeq \mathbb{R}^{d-1}$, $\tau > 0$, and $\sigma > 0$ such that $\tau \geqslant \kappa_0 \sigma$, for κ_0 chosen sufficiently large, using that $\partial\Omega$ is compact.

Thus for any $x \in \partial\Omega$ the local estimate of Theorem 7.4 applies. A similar result applies in the neighborhood of any point of $\Omega \smallsetminus \mathscr{O}_0$.

With the weight function φ constructed above, following the patching procedure described in the proof of Theorem 3.34 in [31], one obtains the following global estimate

$$(8.1) \quad \tau^{-1/2} \| e^{\tau\varphi} u \|_{4,\tau} + | \operatorname{tr}(e^{\tau\varphi} u) |_{3, 1/2, \tau}$$

$$\lesssim \| e^{\tau\varphi} P_\sigma u \|_{L^2(\Omega)} + \sum_{j=1}^{2} | e^{\tau\varphi} B_j u_{|\partial\Omega} |_{7/2 - k_j, \tau} + \tau^{-1/2} \| e^{\tau\varphi} \chi_0 u \|_{4,\tau},$$

for $\tau \geqslant \tau_0$, $\kappa_0 \sigma \leqslant \tau \leqslant \kappa_0' \sigma$, and $u \in \mathscr{C}^\infty(\overline{\Omega})$, and where $\chi_0 \in \mathscr{C}_c^\infty(\mathscr{O})$ such that $\chi_0 \equiv 1$ in a neighborhood of $\overline{\mathscr{O}_1}$. Here, $\|.\|_{s,\tau}$ and $|.|_{s,\tau}$, the Sobolev norms with the large parameter τ, are understood in Ω and $\partial\Omega$ respectively.

REMARK 8.1. — Observe that inequality (8.1) also holds for third-order perturbations of P_σ. Below, we shall use it for a second-order perturbation

$$P_\sigma - i\sigma^2 \alpha = \Delta^2 - \sigma^4 - i\sigma^2 \alpha.$$

8.2. Observability inequality. — By density one finds that inequality 8.1 holds for $u \in H^4(\Omega)$. Let $C_0 > \sup_{\overline{\Omega}} \varphi - 1$. Since $1 \leqslant \varphi \leqslant \sup_{\overline{\Omega}} \varphi$ one obtains

$$(8.2) \qquad \| u \|_{H^4(\Omega)} \lesssim e^{C_0 \tau} \Big(\| P_\sigma u \|_{L^2(\Omega)} + \sum_{j=1}^{2} | B_j u_{|\partial\Omega} |_{H^{7/2 - k_j}(\partial\Omega)} + \| u \|_{H^4(\mathscr{O}_1)} \Big),$$

for $\tau \geqslant \tau_0$, $\kappa_0 \sigma \leqslant \tau \leqslant \kappa_0' \sigma$.

With the ellipticity of P_0 one has

$$\| u \|_{H^4(\mathscr{O}_1)} \lesssim \| P_0 u \|_{L^2(\mathscr{O})} + \| u \|_{L^2(\mathscr{O})},$$

since $\mathscr{O}_1 \Subset \mathscr{O}$. This can be proved by the introduction of a parametrix for P_0. One thus obtain

$$\| u \|_{H^4(\mathscr{O}_1)} \lesssim \| P_\sigma u \|_{L^2(\Omega)} + (1 + \sigma^4) \| u \|_{L^2(\mathscr{O})},$$

and thus with (8.2) one obtains the following observability result.

THEOREM 8.2 (observability inequality). — *Let $P_\sigma = \Delta^2 - \sigma^4$ and let B_1 and B_2 be two boundary operators of order k_1 and k_2 as given in Section* 3.2. *Assume that the Lopatinskiĭ-Šapiro condition of Definition* 3.1 *holds. Let $\mathscr{O}$ be an open set of Ω. There exists $C > 0$ such that*

$$\| u \|_{H^4(\Omega)} \leqslant C e^{C|\sigma|} \Big(\| P_\sigma u \|_{L^2(\Omega)} + \sum_{j=1}^{2} | B_j u_{|\partial\Omega} |_{H^{7/2 - k_j}(\partial\Omega)} + \| u \|_{L^2(\mathscr{O})} \Big),$$

for $u \in H^4(\Omega)$.

Remark 8.3. — With Remark 8.1 the result of Theorem 8.2 hold for $P_\sigma = \Delta^2 - \sigma^4$ replaced by $P_\sigma - i\sigma^2\alpha = \Delta^2 - \sigma^4 - i\sigma^2\alpha$.

9. Solutions to the damped plate equations

Here, we review some aspects of the solutions of the damped plate equation whose form we recall from the introduction:

$$(9.1) \qquad \begin{cases} \partial_t^2 y + Py + \alpha(x)\partial_t y = 0 & (t,x) \in \mathbb{R}_+ \times \Omega, \\ B_1 y_{|\mathbb{R}_+ \times \partial\Omega} = B_2 u_{|\mathbb{R}_+ \times \partial\Omega} = 0, \\ y_{|t=0} = y^0, \quad \partial_t y_{|t=0} = y^1, \end{cases}$$

where $P = \Delta^2$ and $\alpha \geqslant 0$, positive on some open subset of Ω. The boundary operators B_1 and B_2 of orders k_j, $j = 1, 2$, less than or equal to 3 in the normal direction are chosen so that

(i) the Lopatinskiĭ-Šapiro condition of Definition 3.1 is fulfilled for (P, B_1, B_2) on $\partial\Omega$;

(ii) the operator P is symmetric under homogeneous boundary conditions, that is,

$$(9.2) \qquad (Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)},$$

for $u, v \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = B_j v_{|\partial\Omega} = 0$ on $\partial\Omega$, $j = 1, 2$. Examples of such conditions are given in Section 3.5.

With the assumed Lopatinskiĭ-Šapiro condition the operator

$$L : H^4(\Omega) \longrightarrow L^2(\Omega) \oplus H^{7/2-k_1}(\partial\Omega) \oplus H^{7/2-k_2}(\partial\Omega)$$
$$(9.3) \qquad u \longmapsto (Pu, B_1 u_{|\partial\Omega}, B_2 u_{|\partial\Omega})$$

is Fredholm.

(iii) We shall further assume that the Fredholm index of the operator L is zero.

The previous symmetry property gives $(Pu, u)_{L^2(\Omega)} \in \mathbb{R}$. We further assume the following nonnegativity property:

(iv) For $u \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = 0$ on $\partial\Omega$, $j = 1, 2$ one has

$$(9.4) \qquad (Pu, u)_{L^2(\Omega)} \geqslant 0.$$

This last property is very natural to define a nonnegative energy for the plate equation given in (9.1).

We first review some properties of the unbounded operator associated with the bi-Laplace operator and the two homogeneous boundary conditions based on the assumptions made here. Second, the well-posedness of the plate equation is reviewed by means of the a semigroup formulation. This semigroup formalism is also central in the stabilization result in Sections 10.1–10.2.

9.1. THE UNBOUNDED BI-LAPLACE OPERATOR. — Associated with P and the boundary operators B_1 and B_2 is the operator $(\mathsf{P}_0, D(\mathsf{P}_0))$ on $L^2(\Omega)$, with domain

$$D(\mathsf{P}_0) = \big\{ u \in L^2(\Omega) \mid Pu \in L^2(\Omega),\ B_1 u_{|\partial\Omega} = B_2 u_{|\partial\Omega} = 0 \big\},$$

and given by $\mathsf{P}_0 u = Pu \in L^2(\Omega)$ for $u \in D(\mathsf{P}_0)$. The definition of $D(\mathsf{P}_0)$ makes sense since having $Pu \in L^2(\Omega)$ for $u \in L^2(\Omega)$ implies that the traces $\partial_\nu^k u_{|\partial\Omega}$ are well defined for $k = 0, 1, 2, 3$.

Since the Lopatinskiĭ-Šapiro condition holds on $\partial\Omega$ one has $D(\mathsf{P}_0) \subset H^4(\Omega)$ (see for instance Theorem 20.1.7 in [20]) and thus one can also write $D(\mathsf{P}_0)$ as in (1.3). From the assumed nonnegativity in (9.4) above one finds that $\mathsf{P}_0 + \mathrm{Id}$ is injective. Since the operator

$$L' : H^4(\Omega) \longrightarrow L^2(\Omega) \oplus H^{7/2-k_1}(\partial\Omega) \oplus H^{7/2-k_2}(\partial\Omega)$$
$$u \longmapsto (Pu + u, B_1 u_{|\partial\Omega}, B_2 u_{|\partial\Omega})$$

is Fredholm and has the same zero index as L defined in (9.3), one finds that L' is surjective. Thus $\mathrm{Ran}(\mathsf{P}_0 + \mathrm{Id}) = L^2(\Omega)$. One thus concludes that P_0 is maximal monotone. From the assumed symmetry property (9.2) and one finds that P_0 is selfadjoint, using that a symmetric maximal monotone operator is selfadjoint (see for instance Proposition 7.6 in [9]).

The resolvent of $\mathsf{P}_0 + \mathrm{Id}$ being compact on $L^2(\Omega)$, P_0 has a sequence of eigenvalues with finite multiplicities. With the assumed nonnegativity (9.4) they take the form of a sequence

$$0 \leqslant \mu_0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_k \leqslant \cdots$$

that grows to $+\infty$. Associated with this sequence is $(\phi_j)_{j\in\mathbb{N}}$ a Hilbert basis of $L^2(\Omega)$. Any $u \in L^2(\Omega)$ reads $u = \sum_{j\in\mathbb{N}} u_j \phi_j$, with $u_j = (u, \phi_j)_{L^2(\Omega)}$. We define the Sobolev-like scale

$$(9.5) \qquad H_B^k(\Omega) = \{ u \in L^2(\Omega) \mid (\mu_j^{k/4} u_j)_j \in \ell^2(\mathbb{C}) \} \quad \text{for } k \geqslant 0.$$

One has $D(\mathsf{P}_0) = H_B^4(\Omega)$ and $L^2(\Omega) = H_B^0(\Omega)$. Each $H_B^k(\Omega)$, $k \geqslant 0$, is equipped with the inner product and norm

$$(u, v)_{H_B^k(\Omega)} = \sum_{j\in\mathbb{N}} (1 + \mu_j)^{k/2} u_j \overline{v_j}. \qquad \|u\|_{H_B^k(\Omega)}^2 = \sum_{j\in\mathbb{N}} (1 + \mu_j)^{k/2} |u_j|^2,$$

yielding a Hilbert space structure. The space $H_B^k(\Omega)$ is dense in $H_B^{k'}(\Omega)$ if $0 \leqslant k' \leqslant k$ and the injection is compact. Note that one uses $(1 + \mu_j)^{k/2}$ in place of $\mu_j^{k/2}$ since $\ker(\mathsf{P}_0)$ may not be trivial. Note that if $k = 0$ one recovers the standard L^2-inner product and norm.

Using $L^2(\Omega)$ as a pivot space, for $k > 0$ we also define the space $H_B^{-k}(\Omega)$ as the dual space of $H_B^k(\Omega)$. One finds that any $u \in H_B^{-k}(\Omega)$ takes the form of the following limit of L^2-functions

$$u = \lim_{\ell \to \infty} \sum_{j=0}^{\ell} u_j \phi_j,$$

for some $(u_j)_j \subset \mathbb{C}$ such that $\left((1+\mu_j)^{-k/4} u_j \right)_j \in \ell^2(\mathbb{C})$, with the limit occurring in $\left(H_B^k(\Omega) \right)'$ with the natural dual strong topology. Moreover, one has $u_j = \langle u, \overline{\phi_j} \rangle_{H_B^{-k}, H_B^k}$. If $u = \sum_{j \in \mathbb{N}} u_j \phi_j \in H_B^{-k}(\Omega)$ and $v = \sum_{j \in \mathbb{N}} v_j \phi_j \in H_B^k(\Omega)$ one finds

$$\langle u, \overline{v} \rangle_{H_B^{-k}, H_B^k} = \sum_{j \in \mathbb{N}} u_j \overline{v_j}.$$

One can then extend (or restrict) the action of P_0 on any space $H_B^k(\Omega)$, $k \in \mathbb{R}$. One has $\mathsf{P}_0 : H_B^k(\Omega) \to H_B^{k-4}(\Omega)$ continuously with

$$\mathsf{P}_0 u = \sum_{j \in \mathbb{N}} \mu_j u_j \phi_j, \tag{9.6}$$

with convergence in $H_B^{k-4}(\Omega)$ for $u = \sum_{j \in \mathbb{N}} u_j \phi_j \in H_B^k(\Omega)$. In particular, for $u \in H_B^4(\Omega) = D(\mathsf{P}_0)$ and $v \in H_B^2(\Omega)$ one has

$$(\mathsf{P}_0 u, v)_{L^2(\Omega)} = \langle \mathsf{P}_0 u, \overline{v} \rangle_{H_B^{-2}, H_B^2} = \sum_{j \in \mathbb{N}} \mu_j u_j \overline{v_j} \tag{9.7}$$

and if $u, v \in H_B^2(\Omega)$ one has

$$(u, v)_{H_B^2(\Omega)} = (u, v)_{L^2(\Omega)} + \langle \mathsf{P}_0 u, \overline{v} \rangle_{H_B^{-2}, H_B^2} = \sum_{j \in \mathbb{N}} (1 + \mu_j) u_j \overline{v_j}. \tag{9.8}$$

Note that

$$\langle \mathsf{P}_0 u, \overline{v} \rangle_{H_B^{-2}, H_B^2} = (\mathsf{P}_0^{1/2} u, \mathsf{P}_0^{1/2} v)_{L^2(\Omega)}, \tag{9.9}$$

with the operator $\mathsf{P}_0^{1/2}$ easily defined by means of the Hilbert basis $(\phi_j)_{j \in \mathbb{N}}$. In fact, $H_B^2(\Omega)$ is the domain of $\mathsf{P}_0^{1/2}$ viewed as an unbounded operator on $L^2(\Omega)$.

We make the following observations.

(1) If $\ker(\mathsf{P}_0) = \{0\}$ then

$$(u, v) \longmapsto \langle \mathsf{P}_0 u, \overline{u} \rangle_{H_B^{-2}, H_B^2}$$

is also an inner-product on $H_B^2(\Omega)$, that yields an equivalent norm.

(2) If 0 is an eigenvalue, that is, $\dim \ker(\mathsf{P}_0) = n \geqslant 1$ then $(\phi_0, \dots, \phi_{n-1})$ is a orthonormal basis of $\ker(\mathsf{P}_0)$ for the L^2-inner product. From a classical unique continuation property, since $\alpha(x) > 0$ for x in an open subset of Ω one sees that

$$(u, v) \longmapsto (\alpha u, v)_{L^2(\Omega)} \tag{9.10}$$

is also an inner product on the finite dimensional space $\ker(\mathsf{P}_0) \subset L^2(\Omega)$. We introduce a second basis $(\varphi_0, \dots, \varphi_{n-1})$ of $\ker(\mathsf{P}_0)$ orthonormal with respect to this second inner product.

In what follows, we treat the more difficult case where $\dim \ker(\mathsf{P}_0) = n \geqslant 1$. The case $\ker(\mathsf{P}_0) = \{0\}$ is left to the reader.

9.2. **The plate semigroup generator.** — Set $\mathcal{H} = H_B^2(\Omega) \oplus L^2(\Omega)$ with natural inner product and norm

$$(9.11) \qquad \big((u^0, u^1), (v^0, v^1)\big)_{\mathcal{H}} = (u^0, v^0)_{H_B^2(\Omega)} + (u^1, v^1)_{L^2(\Omega)},$$

$$(9.12) \qquad \big\|(u^0, u^1)\big\|_{\mathcal{H}}^2 = \|u^0\|_{H_B^2(\Omega)}^2 + \|u^1\|_{L^2(\Omega)}^2.$$

Define the unbounded operator

$$(9.13) \qquad A = \begin{pmatrix} 0 & -1 \\ \mathsf{P}_0 & \alpha(x) \end{pmatrix},$$

on $\mathcal{H}$ with domain given by $D(A) = D(\mathsf{P}_0) \oplus H_B^2(\Omega)$. This domain is dense in $\mathcal{H}$ and A is a closed operator. One has

$$\mathcal{N} = \ker(A) = \big\{{}^t(u^0, 0) \mid u^0 \in \ker(\mathsf{P}_0)\big\}.$$

The important result of this section is the following proposition.

Proposition 9.1. — *The operator $(A, D(A))$ generates a bounded semigroup $S(t) = e^{-tA}$ on $\mathcal{H}$.*

The understanding of this generator property relies on the introduction of a reduced function space associated with $\ker(\mathsf{P}_0)$, following for instance the analysis of [37]. It will be also important in the derivation of a precise resolvent estimate in Section 10.1. If $\ker(\mathsf{P}_0) = \{0\}$, that is, $\mu_0 > 0$, this procedure is not necessary. For $v \in \ker(\mathsf{P}_0)$, $v \neq 0$, we introduce the linear form

$$(9.14) \qquad \begin{aligned} F_v : \mathcal{H} &\longrightarrow \mathbb{C} \\ (u^0, u^1) &\longmapsto (\alpha v, v)_{L^2(\Omega)}^{-1}\big((\alpha u^0, v)_{L^2(\Omega)} + (u^1, v)_{L^2(\Omega)}\big). \end{aligned}$$

We set

$$(9.15) \qquad \dot{\mathcal{H}} = \bigcap_{\substack{v \in \ker(\mathsf{P}_0) \\ v \neq 0}} \ker(F_v) = \bigcap_{0 \leqslant j \leqslant n-1} \ker(F_{\varphi_j}),$$

with the basis $(\varphi_0, \dots, \varphi_{n-1})$ of $\ker(\mathsf{P}_0)$ introduced above. If $(v, 0) \in \ker(A)$, with $0 \neq v \in \ker(\mathsf{P}_0)$, note that $F_v(v, 0) = 1$. We set $\Theta_j = {}^t(\varphi_j, 0)$, $j = 0, \dots, n-1$ and

$$\Pi_{\mathcal{N}} V = \sum_{j=0}^{n-1} F_{\varphi_j}(V)\Theta_j, \qquad \text{for } V \in \mathcal{H},$$

and $\Pi_{\dot{\mathcal{H}}} = \mathrm{Id}_{\mathcal{H}} - \Pi_{\mathcal{N}}$. We obtain that $\Pi_{\mathcal{N}}$ and $\Pi_{\dot{\mathcal{H}}}$ are *continuous* projectors associated with the direct sum

$$(9.16) \qquad \mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{N} \quad \text{and} \quad \dot{\mathcal{H}} = \ker(\Pi_{\mathcal{N}}).$$

Note that $\dot{\mathcal{H}}$ and $\mathcal{N}$ are *not orthogonal* in $\mathcal{H}$. Yet, it is important to note the following result.

Lemma 9.2. — *We have $\mathrm{Ran}(A) \subset \dot{\mathcal{H}}$.*

Proof. — Let $U = {}^t(u^0, u^1) = AV$ with $V = {}^t(v^0, v^1) \in D(A)$. One has $u^0 = -v^1 \in H^2_B(\Omega)$ and $u^1 = \mathsf{P}_0 v^0 + \alpha v_1 \in L^2(\Omega)$. If $0 \neq \varphi \in \ker(\mathsf{P}_0)$ one writes

$$(\alpha\varphi, \varphi)_{L^2(\Omega)} F_\varphi(U) = (-\alpha v^1, \varphi)_{L^2(\Omega)} + (\mathsf{P}_0 v^0 + \alpha v^1, \varphi)_{L^2(\Omega)}$$
$$= (\mathsf{P}_0 v^0, \varphi)_{L^2(\Omega)} = (v^0, \mathsf{P}_0 \varphi)_{L^2(\Omega)} = 0,$$

using that $v^0, \varphi \in D(\mathsf{P}_0)$, that $(\mathsf{P}_0, D(\mathsf{P}_0))$ is selfadjoint, and that $\varphi \in \ker(\mathsf{P}_0)$. The conclusion follows from the definition of $\dot{\mathcal{H}}$ in (9.15). $\qquad\square$

The space $\dot{\mathcal{H}}$ inherits the natural inner product and norm of $\mathcal{H}$ given in (9.11). Yet one finds that the inner product

$$(9.17) \qquad ((u^0, u^1), (v^0, v^1))_{\dot{\mathcal{H}}} = \langle \mathsf{P}_0 u^0, \overline{v^0} \rangle_{H^{-2}_B, H^2_B} + (u^1, v^1)_{L^2(\Omega)},$$

and associated norm

$$(9.18) \qquad \|(u^0, u^1)\|^2_{\dot{\mathcal{H}}} = \langle \mathsf{P}_0 u^0, \overline{u^0} \rangle_{H^{-2}_B, H^2_B} + \|u^1\|^2_{L^2(\Omega)},$$

yields an equivalent norm on $\dot{\mathcal{H}}$ by a Poincaré-like argument. We introduce the unbounded operator $\dot{A}$ on $\dot{\mathcal{H}}$ given by the domain $D(\dot{A}) = D(A) \cap \dot{\mathcal{H}}$ and such that $\dot{A}V = AV$ for $V \in D(\dot{A})$. We then have $A = \dot{A} \circ \Pi_{\dot{\mathcal{H}}}$. Observe that $D(\dot{A}) = \Pi_{\dot{\mathcal{H}}}(D(A))$ since $\mathcal{N} = \ker(A) \subset D(A)$. Thus, one has

$$(9.19) \qquad D(A) = D(\dot{A}) \oplus \mathcal{N}.$$

As for the decomposition of $\mathcal{H}$ given in (9.16) note that $D(\dot{A})$ and $\mathcal{N}$ are not orthogonal.

Lemma 9.3. — *Let $z \in \mathbb{C}$ be such that $\operatorname{Re} z < 0$. We have*

$$\|(z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A})U\|_{\dot{\mathcal{H}}} \geqslant |\operatorname{Re} z| \, \|U\|_{\dot{\mathcal{H}}}, \quad U \in D(\dot{A}).$$

The proof of this lemma is quite classical. It is given in Appendix A.2. With the previous lemma, with the Hille-Yosida theorem one proves the following result.

Lemma 9.4. — *The operator $(\dot{A}, D(\dot{A}))$ generates a semigroup of contraction $\dot{S}(t) = e^{-t\dot{A}}$ on $\dot{\mathcal{H}}$.*

If we set

$$(9.20) \qquad\qquad S(t) = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} + \Pi_{\mathcal{N}},$$

we find that $S(t)$ is a semigroup on $\mathcal{H}$ generated by $(A, D(A))$, thus proving Proposition 9.1. If $Y^0 \in D(A)$, the solution of the semigroup equation $\frac{d}{dt}Y(t) + AY(t) = 0$ reads

$$(9.21) \qquad\qquad Y(t) = S(t)Y^0 = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} Y^0 + \Pi_{\mathcal{N}} Y^0.$$

We set $\dot{Y}(t) = \Pi_{\dot{\mathcal{H}}} Y(t) = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} Y^0$.

The adjoint of $\dot{A}$ has domain $D(\dot{A}^*) = D(A)$ and is given by

$$\dot{A}^* = \begin{pmatrix} 0 & 1 \\ -\mathsf{P}_0 & \alpha(x) \end{pmatrix}.$$

Similarly to Lemma 9.3 one has the following result with a similar proof.

LEMMA 9.5. — *Let $z \in \mathbb{C}$ be such that $\operatorname{Re} z < 0$. We have*

$$\|(z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}^*)U\|_{\dot{\mathcal{H}}} \geqslant |\operatorname{Re} z| \, \|U\|_{\dot{\mathcal{H}}}, \quad U \in D(\dot{A}^*) = D(\dot{A}).$$

9.3. STRONG AND WEAK SOLUTIONS TO THE DAMPED PLATE EQUATION. — For $y(t)$ a solution to the damped plate equation (9.1) one has $Y(t) = {}^t(y(t), \partial_t y(t))$ formally solution to $\frac{d}{dt}Y(t) + AY(t) = 0$ and conversely.

The semigroup $S(t)$ generated by A as given by Proposition 9.1 allows one to go beyond this formal observation and one obtains the following well-posedness result for strong solutions of the damped plate equation.

PROPOSITION 9.6 (strong solutions of the damped plate equation)

For $(y^0, y^1) \in H_B^4(\Omega) \times H_B^2(\Omega)$ there exists a unique

$$y \in \mathscr{C}^0\big([0, +\infty); H_B^4(\Omega)\big) \cap \mathscr{C}^1\big([0, +\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^2\big([0, +\infty); L^2(\Omega)\big)$$

such that

$$
\text{(9.22)} \qquad
\begin{aligned}
&\partial_t^2 y + Py + \alpha \partial_t y = 0 \quad \text{in } L^\infty([0, +\infty); L^2(\Omega)), \\
&y_{|t=0} = y^0, \ \partial_t y_{|t=0} = y^1.
\end{aligned}
$$

Moreover, there exists $C > 0$ such that

$$\text{(9.23)} \qquad \|y(t)\|_{H_B^4(\Omega)} + \|\partial_t y(t)\|_{H_B^2(\Omega)} \leqslant C\big(\|y^0\|_{H_B^4(\Omega)} + \|y^1\|_{H_B^2(\Omega)}\big), \quad t \geqslant 0.$$

With $Y(t)$ as above, for such a solution $y(t)$ one has

$$\frac{d}{dt}Y(t) + AY(t) = 0, \qquad Y(0) = Y^0 = {}^t(y^0, y^1),$$

that is,

$$Y(t) = S(t)Y^0 \in \mathscr{C}^0\big([0, +\infty); D(A)\big) \cap \mathscr{C}^1\big([0, +\infty); H_B^2(\Omega) \oplus L^2(\Omega)\big).$$

A weak solution to the damped plate equation is simply associated with an initial data $(y^0, y^1) \in H_B^2(\Omega) \times L^2(\Omega)$ and given by the first coordinate of $Y(t) = S(t)Y^0$. Then one has

$$Y(t) \in \mathscr{C}^0\big([0, +\infty); \mathcal{H}\big) \cap \mathscr{C}^1\big([0, +\infty); L^2(\Omega) \oplus H_B^{-2}(\Omega)\big),$$

or equivalently

$$y \in \mathscr{C}^0\big([0, +\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^1\big([0, +\infty); L^2(\Omega)\big) \cap \mathscr{C}^2\big([0, +\infty); H_B^{-2}(\Omega)\big).$$

For a strong solution, the natural energy is given by

$$\text{(9.24)} \qquad \mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|_{L^2(\Omega)}^2 + (\mathsf{P}_0 y(t), y(t))_{L^2(\Omega)}\big).$$

Observe that if $y^0 \in \ker(\mathsf{P}_0)$ then $y(t) = y^0$ is solution to (9.1) with $y^1 = 0$. This is consistent with the form of the semigroup $S(t)$ given in (9.20). Such a solution is independent of the evolution variable t, and thus, despite damping, there is no decay. However, note that such a solution is 'invisible' for the energy defined in (9.24). In fact, for a strong solution to (9.1) as given by Proposition 9.6 one has

$$\text{(9.25)} \qquad \mathcal{E}(y)(t) = \frac{1}{2}\|\dot{Y}(t)\|_{\dot{\mathcal{H}}}^2,$$

with $\dot{Y}(t)$ as defined below (9.21) and $\|.\|_{\dot{\mathcal{H}}}$ defined in (9.18). For a strong solution, we write

$$\frac{d}{dt}\mathcal{E}(y)(t) = \operatorname{Re}(\partial_t y(t), \partial_t^2 y(t))_{L^2(\Omega)} + \frac{1}{2}\langle \mathsf{P}_0 \partial_t y(t), \overline{y(t)}\rangle_{H_B^{-2}, H_B^2} + \frac{1}{2}(\mathsf{P}_0 y(t), \partial_t y(t))_{L^2(\Omega)}$$
$$= \operatorname{Re}(\partial_t y(t), (\partial_t^2 + \mathsf{P}_0)y(t))_{L^2(\Omega)}$$
$$= -\operatorname{Re}(\partial_t y(t), \alpha \partial_t y(t))_{L^2(\Omega)} \leqslant 0$$

since $\alpha \geqslant 0$. Thus, the energy of a strong solution is nonincreasing. To understand the decay of the energy one has to focus on the properties of the semigroup $\dot{S}(t)$ and its generator $(\dot{A}, D(\dot{A}))$ on $\dot{\mathcal{H}}$. This is done in Section 10.1.

For a weak solution $y(t) \in \mathscr{C}^0\big([0, +\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^1\big([0, +\infty); L^2(\Omega)\big)$ the energy is defined by

$$\mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|_{L^2(\Omega)}^2 + \langle \mathsf{P}_0 y(t), \overline{y(t)}\rangle_{H_B^{-2}, H_B^2}\big)$$

that coincides with (9.24) for a strong solution. The stabilization result we are interested in only concerns strong solutions (see Section 10.2). Thus, we shall not mention weak solutions in what follows.

10. Resolvent estimates and applications to stabilization

Here we use the observability inequality of Theorem 8.2 to obtain a resolvent estimate for the plate semigroup generator that allows one to deduce a stabilization result for the damped plate equation. This a sequence of argument comes from the seminal works of Lebeau [35] and Lebeau-Robbiano [37].

10.1. Resolvent estimate. — We prove a resolvent estimate for the unbounded operator $(\dot{A}, D(\dot{A}))$ that acts on $\dot{\mathcal{H}}$. First, we establish that $\{\operatorname{Re} z \leqslant 0\}$ lies in the resolvent set of $\dot{A}$.

Proposition 10.1. — *The spectrum of $(\dot{A}, D(\dot{A}))$ is contained in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$.*

The proof of this proposition is rather classical based on a unique continuation argument and a Fredholm index argument for a compact perturbation. It is given in Appendix A.3.

Theorem 10.2. — *Let $\mathscr{O}$ be an open subset of Ω such that $\alpha \geqslant \delta > 0$ on $\mathscr{O}$. Then, for $\sigma \in \mathbb{R}$ the unbounded operator $i\sigma \operatorname{Id} -\dot{A}$ is invertible on $\dot{\mathcal{H}}$ and for there exist $C > 0$ such that*

$$(10.1) \qquad \|(i\sigma \operatorname{Id} -\dot{A})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} \leqslant C e^{C|\sigma|^{1/2}}, \qquad \sigma \in \mathbb{R}.$$

Proof. — By Proposition 10.1 $i\sigma \operatorname{Id} -\dot{A}$ is indeed invertible. Observe that it then suffices to prove the resolvent estimate (10.1) for $|\sigma| \geqslant \sigma_0$ for some $\sigma_0 > 0$.

Let $U = {}^t(u^0, u^1) \in D(\dot{A})$ and $F = {}^t(f^0, f^1) \in \dot{\mathcal{H}}$ be such that $(i\sigma \operatorname{Id} -\dot{A})U = F$. This reads

$$f^0 = i\sigma u^0 + u^1, \qquad f^1 = -\mathsf{P}_0 u^0 + (i\sigma - \alpha)u^1.$$

which gives
$$(\mathsf{P}_0 - \sigma^2 - i\sigma\alpha)u^0 = f$$
with $f = (i\sigma - \alpha)f^0 - f^1$. Computing the L^2-inner product with u^0 one finds
$$((\mathsf{P}_0 - \sigma^2)u^0, u^0)_{L^2(\Omega)} - i\sigma(\alpha u^0, u^0)_{L^2(\Omega)} = (f, u^0)_{L^2(\Omega)}.$$
As $\alpha \geqslant 0$, computing the imaginary part one obtains
$$\sigma\|\alpha^{1/2}u^0\|_{L^2(\Omega)}^2 = -\operatorname{Im}(f, u^0)_{L^2(\Omega)}.$$
Since $\alpha \geqslant \delta > 0$ in $\mathscr{O}$ by assumption and since we consider $|\sigma| \geqslant \sigma_0$ one has
$$\delta\sigma^0\|u^0\|_{L^2(\mathscr{O})}^2 \leqslant \|f\|_{L^2(\Omega)}\|u^0\|_{L^2(\Omega)}.$$
Applying Theorem 8.2 (with Remark 8.3) one has
$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\big(\|f\|_{L^2(\Omega)} + \|u^0\|_{L^2(\mathscr{O})}\big),$$
replacing $|\sigma|$ by $|\sigma|^2$ therein. Thus, we obtain
$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\big(\|f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}^{1/2}\|u^0\|_{L^2(\Omega)}^{1/2}\big),$$
for $|\sigma| \geqslant \sigma_0$. With Young inequality we write, for $\varepsilon > 0$,
$$e^{C|\sigma|^{1/2}}\|f\|_{L^2(\Omega)}^{1/2}\|u^0\|_{L^2(\Omega)}^{1/2} \lesssim \varepsilon^{-1}e^{2C|\sigma|^{1/2}}\|f\|_{L^2(\Omega)} + \varepsilon\|u^0\|_{L^2(\Omega)}.$$
Thus, with ε chosen sufficiently small one obtains
$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\|f\|_{L^2(\Omega)}.$$
Since $u^1 = f^0 - i\sigma u^0$ and $f = (i\sigma - \alpha)f^0 - f^1$ we finally obtain that
$$\|u^0\|_{H^4(\Omega)} + \|u^1\|_{L^2(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\big(\|f^0\|_{L^2(\Omega)} + \|f^1\|_{L^2(\Omega)}\big)$$
$$\lesssim e^{C|\sigma|^{1/2}}\|F\|_{\dot{\mathcal{H}}}.$$
Since $u^0 \in H^4(\Omega)$ one has
$$\big|(\mathsf{P}_0 u^0, u^0)_{L^2(\Omega)}\big| \leqslant \|u^0\|_{H^4(\Omega)}\|u^0\|_{L^2(\Omega)} \leqslant \|u^0\|_{H^4(\Omega)}^2$$
and thus one finally obtains
$$\|U\|_{\dot{\mathcal{H}}}^2 = (\mathsf{P}_0 u^0, u^0)_{L^2(\Omega)} + \|u^1\|_{L^2(\Omega)}^2 \lesssim e^{C|\sigma|^{1/2}}\|F\|_{\dot{\mathcal{H}}}^2,$$
which concludes the proof of the resolvent estimate (10.1). □

10.2. Stabilization result.
— As an application of the resolvent estimate of Theorem 10.2, we give a logarithmic stabilization result of the damped plate equation (1.1).

For the plate generator $(A, D(A))$ its iterated domains are inductively given by
$$D(A^{n+1}) = \{U \in D(A^n) \mid AU \in D(A^n)\}.$$

With Proposition 9.6, for $Y^0 = {}^t(y^0, y^1) \in D(A^n)$ then the first component of $Y(t) = S(t)Y^0$ is precisely the solution to (9.1). One has $Y(t) = \dot{Y}(t) + \Pi_{\mathcal{N}}Y^0$ with $\dot{Y}(t) = \dot{S}(t)\Pi_{\dot{\mathcal{H}}}Y^0$ with the semigroup $\dot{S}(t)$ defined in Section 9.2. Moreover, by (9.25) the energy of $y(t)$ is given by the square of the $\dot{\mathcal{H}}$-norm of $\dot{Y}(t)$.

With the resolvent estimate of Theorem 10.2, with the result of Theorem 1.5 in [7] one obtains the following bound for the energy of $y(t)$:

$$(10.2) \qquad \mathcal{E}(y)(t) = \|\dot{Y}(t)\|_{\dot{\mathcal{H}}}^2 \leqslant \frac{C}{\big(\log(2+t)\big)^{4n}} \, \|A^n Y^0\|_{\dot{\mathcal{H}}}^2.$$

We have thus obtain the following theorem.

Theorem 10.3 (logarithmic stabilization for the damped plate equation)

Assume that conditions (i) to (iv) of Section 9 hold. Let $n \in \mathbb{N}$, $n \geqslant 1$. Then, there exists $C > 0$ such that for any $Y^0 = {}^t(y^0, y^1) \in D(A^n)$ the associated solution $y(t)$ of the damped plate equation (1.1) has the logarithmic energy decay given by (10.2).

Note that for $n = 1$ using the form of A and (9.9) one recovers the statement of Theorem 1.1 in the introductory section.

Appendix. Some technical results and proofs

A.1. A perfect elliptic estimate. — Here we consider $a(\varrho', \xi_d)$ polynomial in the ξ_d variable and such that its root have negative imaginary parts microlocally.

Lemma A.1. — *Let $\kappa_0 > 0$. Let $a(\varrho', \xi_d) \in S_\tau^{k,0}$, with $\varrho' = (x, \xi', \tau, \sigma)$ and with $k \geqslant 1$, that is, $a(\varrho', \xi_d) = \sum_{j=0}^k a_j(\varrho') \xi_d^{k-j}$, and where the coefficients a_j are homogeneous in (ξ', τ, σ). Moreover, assume that $a_0(\varrho') = 1$. Set $A = \mathrm{Op}(a)$.*

Let $\mathscr{U}$ be a conic open subset of $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geqslant \kappa_0 \sigma$ and such that all the roots of $a(\varrho', \xi_d)$ have a negative imaginary part for $\varrho' \in \mathscr{U}$.

Let $\chi(\varrho') \in S_{\tau,\tau}^0$ be homogeneous of degree zero and such that $\mathrm{supp}(\chi) \subset \mathscr{U}$ and $N \in \mathbb{N}$. Then there exist $C > 0$, $C_N > 0$, and $\tau_0 > 0$ such that

$$\|\mathrm{Op}_\tau(\chi) v\|_{k,\tau} + |\mathrm{tr}(\mathrm{Op}_\tau(\chi) v)|_{k-1,1/2,\tau} \leqslant C \|A\mathrm{Op}_\tau(\chi) v\|_+ + \|v\|_{k,-N,\tau},$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$ and $\tau \geqslant \max(\tau_0, \kappa_0 \sigma)$.

We refer to [8] for a proof (see Lemma 4.1 therein and its proof that adapts to the presence of the parameter σ with $\sigma \lesssim \tau$ in a straightforward manner).

A.2. Basic resolvent estimation. — Here we provide a proof of Lemma 9.3

Let $U = {}^t(u^0, u^1) \in D(\dot{A})$. With (9.17) We write

$$\begin{aligned}
((z \,\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} &= \left(\begin{pmatrix} zu^0 + u^1 \\ zu^1 - \mathsf{P}_0 u^0 - \alpha u^1 \end{pmatrix}, \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \right)_{\dot{\mathcal{H}}} \\
&= z\|U\|_{\dot{\mathcal{H}}}^2 + \langle \mathsf{P}_0 u^1, \overline{u^0} \rangle_{H_B^{-2}, H_B^2} \\
&\qquad\qquad - (\mathsf{P}_0 u^0, u^1)_{L^2(\Omega)} - (\alpha u^1, u^1)_{L^2(\Omega)} \\
&= z\|U\|_{\dot{\mathcal{H}}}^2 + 2i\,\mathrm{Im}(u^1, \mathsf{P}_0 u^0)_{L^2(\Omega)} - (\alpha u^1, u^1)_{L^2(\Omega)}.
\end{aligned}$$

Computing the real part one obtains

$$(A.1) \qquad - \mathrm{Re}((z \,\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} = - \mathrm{Re}(z)\|U\|_{\dot{\mathcal{H}}}^2 + (\alpha u^1, u^1)_{L^2(\Omega)}.$$

As $\alpha \geqslant 0$ and $\operatorname{Re} z < 0$, this gives

$$|\operatorname{Re}((z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}}| \geqslant |\operatorname{Re}(z)| \|U\|_{\dot{\mathcal{H}}}^2,$$

which yields the conclusion of Lemma 9.3. □

A.3. Basic estimation for the resolvent set. — Here we provide a proof of Proposition 10.1. Let $z \in \mathbb{C}$. We consider the two cases.

Case 1: $\operatorname{Re} z < 0$. — By Lemma 9.3 $z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is injective. Moreover, as its adjoint $\overline{z} \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}^*$ is injective and satisfies $\|(\overline{z} \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}^*)U\|_{\dot{\mathcal{H}}} \gtrsim \|U\|_{\dot{\mathcal{H}}}$ for $U \in D(\dot{A})$ by Lemma 9.5 the map $z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is surjective (see for instance [9, Th. 2.20]). The estimation of Lemma 9.3 then gives the continuity of the operator $(z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A})^{-1}$ on $\dot{\mathcal{H}}$.

Case 2: $\operatorname{Re} z = 0$. — We start by proving the injectivity of $z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}$. Let thus $U = {}^t(u^0, u^1) \in D(\dot{A})$ be such that $zU - \dot{A}U = 0$. This gives

(A.2) $$z u^0 + u^1 = 0, \quad -\mathsf{P}_0 u^0 + (z - \alpha) u^1 = 0.$$

First, if $z = 0$ one has $u^1 = 0$, and then $\mathsf{P}_0 u^0 = 0$. Thus, $u^0 \in \ker(\mathsf{P}_0)$ given $U \in \mathcal{N} = \ker(A)$. From the definition of $\dot{\mathcal{H}}$ this gives $U = 0$.

Second, if now $z \neq 0$, using (A.1) we obtain

$$0 = \operatorname{Re}((z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} = -(\alpha u^1, u^1)_{L^2(\Omega)}.$$

As $\alpha \geqslant 0$, this implies that u^0 vanishes a.e. on $\operatorname{supp}(\alpha)$. Observe that

$$\mathsf{P}_0 u^0 = z u^1 = -z^2 u^0.$$

The function u^0 is thus an eigenfunction for P_0 that vanishes on an open set. With the unique continuation property we obtain that u^0 vanishes in Ω and u^1 as well.

If we now prove that $z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is surjective, the result then follows from the closed graph theorem as $\dot{A}$ is a closed operator. We write $z \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A} = T + \operatorname{Id}_{\dot{\mathcal{H}}}$ with $T = (z-1) \operatorname{Id}_{\dot{\mathcal{H}}} - \dot{A}$. By the first part of the proof, T is invertible with a bounded inverse. The operator T is unbounded on $\dot{\mathcal{H}}$. We denote by $\widetilde{T}$ the restriction of T to $D(\dot{A})$ equipped with the graph-norm associated with $\dot{A}$. The operator $\widetilde{T}$ is bounded. It is also invertible. It is thus a bounded Fredholm operator of index $\operatorname{ind} \widetilde{T} = 0$. Similarly, we denote by ι the injection of $D(\dot{A})$ into $\dot{\mathcal{H}}$ and $\widetilde{A}$ the restriction of $\dot{A}$ on $D(\dot{A})$ viewed as a bounded operator. We have $z\iota - \widetilde{A} = \widetilde{T} + \iota$. Since ι is a compact operator, we obtain that $z\iota - \widetilde{A}$ is also a bounded Fredholm operator of index 0. Hence, $z\iota - \widetilde{A}$ is surjective since $z \operatorname{Id}_{\mathcal{H}} - \dot{A}$ is injective as proved above. Consequently, $z \operatorname{Id}_{\mathcal{H}} - \dot{A}$ is surjective. This concludes the proof of Proposition 10.1. □

References

[1] F. Alabau-Boussouira – "Piecewise multiplier method and nonlinear integral inequalities for Petrowsky equation with nonlinear dissipation", *J. Evol. Equ.* **6** (2006), no. 1, p. 95–112.

[2] F. Alabau-Boussouira & K. Ammari – "Sharp energy estimates for nonlinearly locally damped PDEs via observability for the associated undamped system", *J. Funct. Anal.* **260** (2011), no. 8, p. 2424–2450.

[3] F. Alabau-Boussouira, Y. Privat & E. Trélat – "Nonlinear damped partial differential equations and their uniform discretizations", *J. Funct. Anal.* **273** (2017), no. 1, p. 352–403.

[4] K. Ammari, M. Tucsnak & G. Tenenbaum – "A sharp geometric condition for the boundary exponential stabilizability of a square plate by moment feedbacks only", in *Control of coupled partial differential equations*, Internat. Ser. Numer. Math., vol. 155, Birkhäuser, Basel, 2007, p. 1–11.

[5] V. Barbu – "Exact controllability of the superlinear heat equation", *Appl. Math. Optim.* **42** (2000), no. 1, p. 73–89.

[6] C. Bardos, G. Lebeau & J. Rauch – "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30** (1992), no. 5, p. 1024–1065.

[7] C. J. K. Batty & T. Duyckaerts – "Non-uniform stability for bounded semi-groups on Banach spaces", *J. Evol. Equ.* **8** (2008), no. 4, p. 765–780.

[8] M. Bellassoued & J. Le Rousseau – "Carleman estimates for elliptic operators with complex coefficients. Part I: Boundary value problems", *J. Math. Pures Appl. (9)* **104** (2015), no. 4, p. 657–728.

[9] H. Brezis – *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[10] A. L. Bukhgeĭm & M. V. Klibanov – "Global uniqueness of a class of multidimensional inverse problems", *Soviet Math. Dokl.* **24** (1981), p. 244–247.

[11] A.-P. Calderón – "Uniqueness in the Cauchy problem for partial differential equations", *Amer. J. Math.* **80** (1958), p. 16–36.

[12] T. Carleman – "Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes.", *Ark. Mat. Astron. Fys.* **26** (1939), no. 17, p. 1–9.

[13] R. Denk & R. Schnaubelt – "A structurally damped plate equation with Dirichlet-Neumann boundary conditions", *J. Differential Equations* **259** (2015), no. 4, p. 1323–1353.

[14] D. Dos Santos Ferreira – "Sharp L^p Carleman estimates and unique continuation", *Duke Math. J.* **129** (2005), no. 3, p. 503–550.

[15] D. Dos Santos Ferreira, C. E. Kenig, M. Salo & G. Uhlmann – "Limiting Carleman weights and anisotropic inverse problems", *Invent. Math.* **178** (2009), no. 1, p. 119–171.

[16] E. Fernández-Cara & E. Zuazua – "Null and approximate controllability for weakly blowing up semilinear heat equations", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **17** (2000), no. 5, p. 583–616.

[17] A. V. Fursikov & O. Y. Imanuvilov – *Controllability of evolution equations*, Lect. Notes Series, vol. 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[18] L. Hörmander – "On the uniqueness of the Cauchy problem", *Math. Scand.* **6** (1958), p. 213–225.

[19] _____, *Linear partial differential operators*, Grundlehren Math. Wiss., vol. 116, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[20] _____, *The analysis of linear partial differential operators.* III, Classics in Math., Springer, Berlin, 2007.

[21] O. Imanuvilov, V. Isakov & M. Yamamoto – "An inverse problem for the dynamical Lamé system with two sets of boundary data", *Comm. Pure Appl. Math.* **56** (2003), no. 9, p. 1366–1382.

[22] V. Isakov – *Inverse problems for partial differential equations*, third ed., Applied Math. Sciences, vol. 127, Springer, Cham, 2017.

[23] S. Jaffard – "Contrôle interne exact des vibrations d'une plaque rectangulaire", *Portugal. Math.* **47** (1990), no. 4, p. 423–429.

[24] D. Jerison & C. E. Kenig – "Unique continuation and absence of positive eigenvalues for Schrödinger operators", *Ann. of Math. (2)* **121** (1985), no. 3, p. 463–494, With an appendix by E. M. Stein.

[25] J.-P. Kahane – "Pseudo-périodicité et séries de Fourier lacunaires", *Ann. Sci. École Norm. Sup.* **79** (1962), p. 93–150.

[26] C. E. Kenig, J. Sjöstrand & G. Uhlmann – "The Calderón problem with partial data", *Ann. of Math. (2)* **165** (2007), no. 2, p. 567–591.

[27] H. Koch & D. Tataru – "Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients", *Comm. Pure Appl. Math.* **54** (2001), no. 3, p. 339–360.

[28] _____, "Sharp counterexamples in unique continuation for second order elliptic equations", *J. reine angew. Math.* **542** (2002), p. 133–146.

[29] _____ , "Dispersive estimates for principally normal pseudodifferential operators", *Comm. Pure Appl. Math.* **58** (2005), no. 2, p. 217–284.

[30] M. Kubo – "Uniqueness in inverse hyperbolic problems—Carleman estimate for boundary value problems", *J. Math. Kyoto Univ.* **40** (2000), no. 3, p. 451–473.

[31] J. Le Rousseau, G. Lebeau & L. Robbiano – *Elliptic Carleman estimates and applications to stabilization and controllability. Vol. I. Dirichlet boundary conditions on Euclidean space*, Progress in Nonlinear Differential Equations and their Applications, vol. 97, Birkhäuser/Springer, Cham, 2022.

[32] _____ , *Elliptic Carleman estimates and applications to stabilization and controllability. Vol. II. General boundary conditions on Riemannian manifolds*, Progress in Nonlinear Differential Equations and their Applications, vol. 98, Birkhäuser/Springer, Cham, 2022.

[33] J. Le Rousseau & L. Robbiano – "Spectral inequality and resolvent estimate for the bi-Laplace operator", *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 4, p. 1003–1094.

[34] G. Lebeau – "Contrôle de l'équation de Schrödinger", *J. Math. Pures Appl. (9)* **71** (1992), no. 3, p. 267–291.

[35] _____ , "Équation des ondes amorties", in *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, Math. Phys. Stud., vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, p. 73–109.

[36] G. Lebeau & L. Robbiano – "Contrôle exact de l'équation de la chaleur", *Comm. Partial Differential Equations* **20** (1995), no. 1-2, p. 335–356.

[37] G. Lebeau & L. Robbiano – "Stabilisation de l'équation des ondes par le bord", *Duke Math. J.* **86** (1997), no. 3, p. 465–491.

[38] K. Ramdani, T. Takahashi & M. Tucsnak – "Internal stabilization of the plate equation in a square: the continuous and the semi-discretized problems", *J. Math. Pures Appl. (9)* **85** (2006), no. 1, p. 17–37.

[39] J. Rauch & M. Taylor – "Exponential decay of solutions to hyperbolic equations in bounded domains", *Indiana Univ. Math. J.* **24** (1974), p. 79–86.

[40] C. D. Sogge – "Oscillatory integrals and unique continuation for second order elliptic differential equations", *J. Amer. Math. Soc.* **2** (1989), no. 3, p. 491–515.

[41] L. Tebou – "Well-posedness and stability of a hinged plate equation with a localized nonlinear structural damping", *Nonlinear Anal.* **71** (2009), no. 12, p. e2288–e2297.

[42] _____ , "Well-posedness and stabilization of an Euler-Bernoulli equation with a localized nonlinear dissipation involving the *p*-Laplacian", *Discrete Contin. Dynam. Systems* **32** (2012), no. 6, p. 2315–2337.

[43] C. Zuily – *Uniqueness and non-uniqueness in the Cauchy problem*, Progress in Math., vol. 33, Birkhäuser, Cham, 1983.

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