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TWISTED LIMIT FORMULA FOR TORSION AND CYCLIC BASE CHANGE

by Nicolas Bergeron & Michael Lipnowski

Abstract. — Let G be the group of complex points of a real semi-simple Lie group whose fundamental rank is equal to 1, e.g. $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_3(\mathbb{C})$. Then the fundamental rank of G is 2, and according to the conjecture made in [3], lattices in G should have 'little' — in the very weak sense of 'subexponential in the co-volume' — torsion homology. Using base change, we exhibit sequences of lattices where the torsion homology grows exponentially with the *square root* of the volume. This is deduced from a general theorem that compares twisted and untwisted L^2-torsions in the general base-change situation. This also makes uses of a precise equivariant 'Cheeger-Müller Theorem' proved by the second author [23].

Résumé (Formule de multiplicité limite tordue pour la torsion et changement de base cyclique)
Soit G le groupe des points complexes d'un groupe de Lie semi-simple réel dont le rang fondamental est égal à 1, par exemple $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ ou $\mathrm{SL}_3(\mathbb{C})$. Alors le rang fondamental de G est égal à 2 et, selon la conjecture faite dans [3], les réseaux dans G devraient avoir « peu » — dans le sens très faible de « sous-exponentiel en le co-volume » — de torsion homologique. En utilisant le changement de base, nous exhibons des suites de réseaux le long desquelles la torsion homologique croît exponentiellement avec la racine carrée du volume. Ce comportement est déduit d'un théorème général qui compare les torsions L^2 tordues et non tordues dans la situation générale d'un changement de base. Nous utilisons également une version équivariante précise du « Théorème de Cheeger-Müller » démontrée par le second auteur [23].

Contents

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1. Introduction

1.1. Asymptotic growth of cohomology. — Let Γ be a torsion-free uniform lattice in a semisimple Lie group G with maximal compact subgroup K. Let $\Gamma_n \subset \Gamma$ be a decreasing sequence of normal subgroups with trivial intersection. It is known that

$$\lim_{n\to\infty} \frac{\dim H^j(\Gamma_n, \mathbb{C})}{[\Gamma : \Gamma_n]}$$

converges to $b_j^{(2)}(\Gamma)$, the jth L^2-Betti number of Γ. If $b_j^{(2)} \neq 0$ for some j, it follows that cohomology is abundant. However, it is often true that $b_j^{(2)}(\Gamma) = 0$ for all j; this is the case whenever $\delta(G) := \operatorname{rank}_{\mathbb{C}} G - \operatorname{rank}_{\mathbb{C}} K \neq 0$. What is the true rate of growth of $b_j(\Gamma_n) = \dim H^j(\Gamma_n, \mathbb{C})$ when $\delta(G) \neq 0$? In particular, is $b_j(\Gamma_n)$ non-zero for sufficiently large n?

We address this question for 'cyclic base-change.' Before stating a general result, let's give two typical examples of this situation.

Examples

(1) The real semisimple Lie group $G = \mathrm{SL}_2(\mathbb{C})$ satisfies $\delta = 1$. Let $\sigma : G \to G$ be the real involution given by complex conjugation.

(2) The real semisimple Lie group $G = \mathrm{SL}_2(\mathbb{C}) \times \cdots \times \mathrm{SL}_2(\mathbb{C})$ (n times) satisfies $\delta = n$. Let $\sigma : G \to G$ be the order $2n$ automorphism of G given by $\sigma(g_1, \ldots, g_n) = (\overline{g}_n, g_1, \ldots, g_{n-1})$.

Now let $\Gamma_n \subset \Gamma$ be a sequence of finite index, σ-stable subgroups of G. It follows from the general Proposition 1.2 below that

$$\sum_j b_j(\Gamma_n) \gg \mathrm{vol}(\Gamma_n^\sigma \backslash \mathrm{SL}_2(\mathbb{R})).$$

Note that when the Γ_n's are congruence subgroups of an arithmetic lattice Γ, then $\mathrm{vol}(\Gamma_n^\sigma \backslash \mathrm{SL}_2(\mathbb{R}))$ grows like $\mathrm{vol}(\Gamma_n \backslash G)^{1/\mathrm{order}(\sigma)}$.

In this paper, we shall more generally consider the case where G is obtained from a real algebraic group by 'base change.' Let $\mathbf{G}$ be a connected semisimple quasi-split algebraic group defined over $\mathbb{R}$. Let $\mathbb{E}$ be an étale $\mathbb{R}$-algebra such that $\mathbb{E}/\mathbb{R}$ is a cyclic Galois extension with Galois group generated by $\sigma \in \mathrm{Aut}(\mathbb{E}/\mathbb{R})$. Concretely, $\mathbb{E}$ is either $\mathbb{R}^s$ or $\mathbb{C}^s$. In the first case σ is of order s and acts on $\mathbb{R}^s$ by cyclic permutation. In the second case σ is of order $2s$ and acts on $\mathbb{C}^s$ by $(z_1, \ldots, z_s) \mapsto (\overline{z}_s, z_1, \ldots, z_{s-1})$. The automorphism σ induces a corresponding automorphism of the group G of real points of $\mathrm{Res}_{\mathbb{E}/\mathbb{R}} \mathbf{G}$.(1) We will furthermore assume that $H^1(\sigma, G) = \{1\}$; see Section 2.4 for comments on this condition. The following proposition is 'folklore' (see e.g. Borel-Labesse-Schwermer [6], Rohlfs-Speh [30] and Delorme [14]).

(1)The assumption that $\mathbf{G}/\mathbb{R}$ is quasi-split is used in the case where $\mathbb{E} = \mathbb{C}$, where we quote results of Delorme [14] concerning base change from $\mathbf{G}(\mathbb{R})$ to $\mathbf{G}(\mathbb{C})$. We emphasize that assuming $\mathbf{G}/\mathbb{R}$ is quasi-split is unnecessary in the case where $\mathbb{E} = \mathbb{R}^s$ and σ acts by cyclic permutation. See Section 5.

1.2. Proposition. — *Let $\Gamma_n \subset \Gamma$ be a sequence of finite index, σ-stable subgroups of G. Suppose that $\delta(G^\sigma)=0$. Then we have:*

$$\sum_j \dim b_j(\Gamma_n) \gg \mathrm{vol}(\Gamma_n^\sigma \backslash G^\sigma).$$

We prove Proposition 1.2 for certain families $\{\Gamma_n\}$ in Section 4.6 but our real interest here is rather how the *torsion cohomology* grows.

1.3. Asymptotic growth of torsion cohomology. — Let (ρ, F) be a finite dimensional representation of G defined over $\mathbb{R}$ and suppose that the Γ_n's stabilize some fixed lattice $\mathscr{O} \subset F$. The first named author and Venkatesh [3] prove that for 'strongly acyclic' [3, §4] representations ρ, there is a lower bound

$$\sum_j \log|H^j(\Gamma_n, \mathscr{O})_{\mathrm{tors}}| \gg c(G,\rho)\cdot[\Gamma:\Gamma_n]$$

for some constant $c(G,\rho)$. In fact, they prove a limiting identity

$$\frac{\sum_j (-1)^j \log|H^j(\Gamma_n, \mathscr{O})_{\mathrm{tors}}|}{[\Gamma:\Gamma_n]} \longrightarrow c(G,\rho) \tag{1.3.1}$$

and prove that $c(G,\rho)$ is non-zero exactly when $\delta(G)=1$. The numerator of the left side of (1.3.1) should be thought of as a 'torsion Euler characteristic.' The purpose of this article is to prove an analogous theorem about 'torsion Lefschetz numbers.'

To state one instance of our main result, we keep assuming that $H^1(\sigma, G)=\{1\}$ and furthermore assume that:

(1) σ has prime order p and $\mathscr{O}_{\mathbb{F}_p}$ is trivial.

(2) the Γ_n's are σ-stable finite index subgroups of Γ such that $\bigcap_n \Gamma_n = \{1\}$ (or, more generally, that satisfy the hypothesis of Section 4.3), and

(3) the representation ρ is strongly acyclic and can be extended to a finite dimensional (twisted) representation $\widetilde{\rho}$ of the twisted space $\widetilde{G} = G \rtimes \sigma$ that is *strongly acyclic* (see Section 2.6).

Under these hypotheses we shall prove the following:

1.4. Theorem. — *We have:*

$$\limsup \frac{\sum_j \log|H^j(\Gamma_n, \mathscr{O})|}{\mathrm{vol}(\Gamma_n^\sigma \backslash G^\sigma)} > 0 \tag{1.4.1}$$

whenever $\delta(G^\sigma)=1$.

Example. — Let $E/\mathbb{Q}$ be an imaginary quadratic extension. Let B be a division algebra of dimension 9 over $\mathbb{Q}$ such that B is split at infinity and $B_E := B\otimes_{\mathbb{Q}} E$ is a division algebra. Let $\mathfrak{o}$ be a maximal order in B and let $\mathfrak{o}_E$ be its tensor product over $\mathbb{Z}$ with the ring of integers of E. Then $\mathfrak{o}_E^\times$ embeds into $\mathrm{PGL}_3(\mathbb{C})$. Let $\mathscr{O}$ be the set of elements in $\mathfrak{o}_E$ of trace 0, considered as an $\mathfrak{o}_E^\times$-module by conjugation. Let Γ be a torsion free congruence subgroup of $\mathfrak{o}_E^\times$ such that Γ is contained in the kernel of ρ mod 2. Then the local system $\mathscr{O}_{\mathbb{F}_2}$ is trivial. Given a prime q, we denote by Γ_q the kernel of the reduction map $\Gamma \mapsto (\mathfrak{o}_E/q\mathfrak{o}_E)^\times$. Theorem 1.4 applies to the sequence $\{\Gamma_q\}$ and we conclude that

$$\limsup \frac{1}{q^8} \sum_j \log |H^j(\Gamma_q, \mathscr{O})| > 0. \tag{1.4.2}$$

Here, q^8 is the growth rate of the log of torsion in H^3 of the corresponding q-congruence subgroups of $\mathfrak{o}^\times$ embedded into $\mathrm{PGL}_3(\mathbb{R})$. The cohomology classes that contribute to (1.4.2) should conjecturally arise by base change transfer over $\mathbb{Z}$. One may regard (1.4.2) as a partial evidence for the existence of such a transfer.

Similarly, one can construct examples of lattices Γ in $\mathrm{SL}_2(\mathbb{C})^p$ ($p > 1$ prime), in $\mathrm{SL}_3(\mathbb{C})$ or in $\mathrm{SL}_4(\mathbb{C})$ such that the torsion homology of level q congruence subgroups of Γ grows exponentially with respectively q^6, q^8 or q^{15}.

1.5. — Analogously to (1.3.1), Equation (1.4.1) follows from a limiting identity for torsion Lefschetz numbers. For example, when $\sigma^2 = 1$, conditional on an assumption about the growth of the Betti numbers $\dim_{\mathbb{F}_2} H^j(\Gamma_n^\sigma, \mathscr{O}_{\mathbb{F}_2})$ we prove that

$$\frac{\sum (-1)^j (\log |H^j(\Gamma_n, \mathscr{O})^+_{\mathrm{tors}}| - \log |H^j(\Gamma_n, \mathscr{O})^-_{\mathrm{tors}}|)}{|H^1(\sigma, \Gamma_n)| \operatorname{vol}(\Gamma_n^\sigma \backslash G^\sigma)} \longrightarrow c(G, \rho, \sigma), \tag{1.5.1}$$

where the superscript $\pm$ denotes the ± 1 eigenspace.

Assume that the maximal compact subgroup $K \subset G$ is σ-stable and let $X = G/K$ and $X^\sigma = G^\sigma / K^\sigma$. The proof of (1.5.1) crucially uses the equivariant Cheeger-Müller theorem, proven by Bismut-Zhang [5]. This enables us to compute the left side of (1.5.1) (up to a controlled integer multiple of $\log 2$) by studying the eigenspaces of the Laplace operators of the metrized local system associated to ρ together with their σ action. More precisely, the left side of (1.5.1) nearly equals the equivariant analytic torsion $\log T^\sigma_{\Gamma_n \backslash X}(\rho)$; see (3.5.1) for a definition of the latter. Using the simple twisted trace formula and results of Bouaziz [7], we prove a 'limit multiplicity formula.'

1.6. Theorem. — *Assume* (1)–(3) *above. Then we have:*

$$\frac{\log T^\sigma_{\Gamma_n \backslash X}(\rho)}{|H^1(\sigma, \Gamma_n)| \operatorname{vol}(\Gamma_n^\sigma \backslash G^\sigma)} \longrightarrow s 2^r t^{(2)}_{X^\sigma}(\rho), \tag{1.6.1}$$

where $\mathbb{E} = \mathbb{R}^s$ *or* $\mathbb{C}^s$, *and* $r = 0$ *in the first case and* $r = \operatorname{rank}_{\mathbb{R}} \mathbf{G}(\mathbb{C}) - \operatorname{rank}_{\mathbb{R}} \mathbf{G}(\mathbb{R})$ *in the second case.*

Here, $t^{(2)}_{X^\sigma}(\rho)$ is the (usual) L^2-analytic torsion of the symmetric space X^σ twisted by the finite dimensional representation ρ. It is explicitly computed in [3]. Note that it is non-zero if and only if $\delta(G^\sigma) = 1$.

The authors hope that the limit multiplicity formula (1.6.1) together with the twisted endoscopic comparison implicit in Section 7 will be of interest independent of torsion in cohomology. These computations complement work by Borel-Labesse-Schwermer [6] and Rohlfs-Speh [30].

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2. The simple twisted trace formula

Let $\mathbf{G}$ be a connected semisimple quasi-split algebraic group defined over $\mathbb{R}$. Let $\mathbb{E}$ be an étale $\mathbb{R}$-algebra such that $\mathbb{E}/\mathbb{R}$ is a cyclic Galois extension with Galois group generated by $\sigma \in \mathrm{Aut}(\mathbb{E}/\mathbb{R})$. The automorphism σ induces a corresponding automorphism of the group G of real points of $\mathrm{Res}_{\mathbb{E}/\mathbb{R}}\,\mathbf{G}$. We furthermore choose a Cartan involution θ of G that commutes with σ and denote by K the group of fixed points of θ in G. Here we follow Labesse-Waldspurger [20].

2.1. Twisted spaces. — We associate to these data the *twisted space*

$$\widetilde{G} = G \rtimes \sigma \subset G \rtimes \mathrm{Aut}(G).$$

The left action of G on G,

$$(g, x \rtimes \sigma) \longmapsto gx \rtimes \sigma,$$

turns $\widetilde{G}$ into a left principal homogeneous G-space equipped with a G-equivariant map $\mathrm{Ad} : \widetilde{G} \to \mathrm{Aut}(G)$ given by

$$\mathrm{Ad}(x \rtimes \sigma)(g) = \mathrm{Ad}(x)(\sigma(g)).$$

We also have a right action of G on $\widetilde{G}$ by

$$\delta g = \mathrm{Ad}(\delta)(g)\delta \quad (\delta \in \widetilde{G},\ g \in G).$$

This enables us to define an action by conjugation of G on $\widetilde{G}$ and yields a notion of G-conjugacy class in $\widetilde{G}$. The left and right actions of G on the twisted space $G \rtimes \sigma$ are induced by the left and right multiplications in the semi-direct product $G \rtimes \langle\sigma\rangle$. In particular, taking $\delta = 1 \rtimes \sigma$ we have:

$$(1 \rtimes \sigma)g = \delta g = \sigma(g)\delta = \sigma(g) \rtimes \sigma.$$

We similarly define the twisted space $\widetilde{K} = K \rtimes \sigma$.

2.2. Twisted representations. — A *representation of* $\widetilde{G}$, in a vector space V, is the data for every $\delta \in \widetilde{G}$ of a invertible linear map

$$\widetilde{\pi}(\delta) \in \mathrm{GL}(V)$$

and of a representation of G in V:

$$\pi : G \longrightarrow \mathrm{GL}(V)$$

such that for $x, y \in G$ and $\delta \in \widetilde{G}$,

$$\widetilde{\pi}(x\delta y) = \pi(x)\widetilde{\pi}(\delta)\pi(y).$$

In particular

$$\widetilde{\pi}(\delta x) = \pi(\mathrm{Ad}(\delta)(x))\widetilde{\pi}(\delta).$$

Therefore $\widetilde{\pi}(\delta)$ intertwines π and $\pi \circ \mathrm{Ad}(\delta)$. Note that $\widetilde{\pi}$ is the restriction to $G \rtimes \sigma$ of a genuine representation of $G \rtimes \langle \sigma \rangle$. As such it determines π; we will say that π is the restriction of $\widetilde{\pi}$ to G.

Conversely $\widetilde{\pi}$ is determined by the data of π and of an operator A which intertwines π and $\pi \circ \sigma$:

$$A\pi(x) = (\pi \circ \sigma)(x)A$$

and whose p-th power is the identity, where p is the order of σ. We reconstruct $\widetilde{\pi}$ by setting

$$\widetilde{\pi}(x \rtimes \sigma) = \pi(x)A \quad \text{for } x \in G.$$

Say that $\widetilde{\pi}$ is *essential* if π is irreducible. If $\widetilde{\pi}$ is unitary and essential, Schur's lemma implies that π determines A up to a p-th root of unity.

There is a natural notion of equivalence between representations of $\widetilde{G}$ — see e.g. [20, §2.3]. This is the obvious one; beware however that, even if $\widetilde{\pi}$ is essential, the class of π does not determine the class of $\widetilde{\pi}$ since the intertwiner A is only determined up to a root of unity. We have a corresponding notion of a $(\mathfrak{g}, \widetilde{K})$-module.

If $\widetilde{\pi}$ is unitary and $f \in C_c^\infty(\widetilde{G})$ we set

$$\widetilde{\pi}(f) = \int_{\widetilde{G}} f(y)\widetilde{\pi}(y)\,dy := \int_G f(x \rtimes \sigma)\widetilde{\pi}(x \rtimes \sigma)\,dx.$$

It follows from [20, Lem. 2.3.2] that $\widetilde{\pi}(f)$ is of trace class. Moreover: $\operatorname{trace}\widetilde{\pi}(f) = 0$ unless $\widetilde{\pi}$ is essential. In the following, we denote by $\Pi(\widetilde{G})$ the set of irreducible unitary representations π of G (considered up to equivalence) that can be extended to some (twisted) representation $\widetilde{\pi}$ of $\widetilde{G}$. Note that the extension is not unique.

2.3. Twisted trace formula (in the cocompact case). — Let Γ be a cocompact lattice of G that is σ-stable. Associated to Γ is the (right) regular representation $\widetilde{R}_\Gamma$ of $\widetilde{G}$ on $L^2(\Gamma\backslash G)$, where the restriction R_Γ of $\widetilde{R}_\Gamma$ is the usual regular representation in $L^2(\Gamma\backslash G)$ and

$$(\widetilde{R}_\Gamma(\sigma))(f)(\Gamma x) = f(\Gamma\sigma(x)).$$

Note that

$$(\widetilde{R}_\Gamma(\sigma)R_\Gamma(g))(f)(\Gamma x) = f(\Gamma\sigma(x)g) = (R_\Gamma(\sigma(g))\widetilde{R}_\Gamma(\sigma))(f)(\Gamma x).$$

Given $\delta \in \widetilde{G}$ we denote by G^δ its centralizer in G (for the (twisted) action by conjugation of G on $\widetilde{G}$). Corresponding to Γ is a (non-empty) discrete twisted subspace $\widetilde{\Gamma} \subset \widetilde{G}$. Given $\delta \in \widetilde{\Gamma}$ we denote by $\{\delta\}$ its Γ-conjugacy class (where here again Γ acts by (twisted) conjugation on $\widetilde{\Gamma}$).

Let $f \in C_c^\infty(\widetilde{G})$. The twisted trace formula is obtained by computing the trace of $\widetilde{R}_\Gamma(f)$ in two different ways. It takes the following form (the LHS is the spectral side and the RHS is the geometric side):

$$\sum_{\pi \in \Pi(\widetilde{G})} m(\pi, \widetilde{\pi}, \Gamma)\operatorname{trace}\widetilde{\pi}(f) = \sum_{\{\delta\}} \mathrm{vol}(\Gamma^\delta\backslash G^\delta) \int_{G^\delta\backslash G} f(x^{-1}\delta x)\,d\dot{x}. \tag{2.3.1}$$

Here, $\widetilde{\pi}$ is some extension of π to a twisted representation of $\widetilde{G}$ and

$$\begin{aligned} m(\pi,\widetilde{\pi},\Gamma) &= \sum_{\widetilde{\pi}'|G=\pi} \lambda(\widetilde{\pi}',\widetilde{\pi}) m(\widetilde{\pi}') \\ &= \operatorname{trace}\bigl(\sigma| \operatorname{Hom}_G(\widetilde{\pi}, L^2(\Gamma\backslash G))\bigr), \end{aligned}$$

where $m(\widetilde{\pi}')$ is the multiplicity of $\widetilde{\pi}'$ in $\widetilde{R}_\Gamma$ and $\lambda(\widetilde{\pi}',\widetilde{\pi}) \in \mathbb{C}^\times$ is the scalar such that, for all $\delta \in \widetilde{G}$, we have $\widetilde{\pi}'(\delta) = \lambda(\widetilde{\pi}',\widetilde{\pi})\widetilde{\pi}(\delta)$.(2) Note that $\lambda(\widetilde{\pi}',\widetilde{\pi})$ is in fact a p-th root of unity.

The definition of $\operatorname{trace}\widetilde{\pi}(f)$ depends on a choice of a Haar measure dx on G. On the geometric side the volumes $\operatorname{vol}(\Gamma^\delta\backslash G^\delta)$ depend on choices of Haar measures on the groups G^δ. We will make precise choices later on. For the moment we just note that the measure $d\dot{x}$ on the quotient $G^\delta\backslash G$ is normalized by:

$$\int_G \phi(x)\,dx = \int_{G^\delta\backslash G}\int_{G^\delta} \phi(gx)\,dg\,d\dot{x}.$$

2.4. Galois cohomology groups $H^1(\sigma,\Gamma)$. — Let $Z^1(\sigma,\Gamma) = \{\delta \in \widetilde{\Gamma} \,:\, \delta^p = 1\}$; it is invariant by conjugation by Γ. We denote by $H^1(\sigma,\Gamma)$ the quotient of $Z^1(\sigma,\Gamma)$ by the equivalence relation defined by conjugation by elements of Γ. We have similar definitions when Γ is replaced by G.

We will assume that

$$H^1(\sigma, G) = \{1\}. \tag{2.4.1}$$

Note that the set $H^1(\sigma,G)$ is the same as the nonabelian cohomology group $H^1(\langle\sigma\rangle,\Gamma)$ or $H^1(\mathbb{E}/\mathbb{R},\Gamma)$ if $\mathbb{E}/\mathbb{R}$ is a cyclic Galois extension with Galois group generated by $\sigma \in \operatorname{Aut}(\mathbb{E}/\mathbb{R})$.

If $\mathbb{E} = \mathbb{R}^p$, Condition (2.4.1) is always satisfied. Indeed: in that case $G = \mathbf{G}(\mathbb{R})^p$ and an element

$$(g_1,\ldots,g_p) \rtimes \sigma \in \widetilde{G}$$

belongs to $Z^1(\sigma,G)$ if and only if $g_1g_2\cdots g_p = e$. But then there is an equality

$$\sigma(g_1, g_1g_2, \ldots, g_1\cdots g_p)^{-1}(g_1, g_1g_2,\ldots,g_1\cdots g_p) = (g_1,\ldots,g_p).$$

Equivalently, $(g_1,\ldots,g_p)\rtimes\sigma$ is conjugated to σ in $\widetilde{G}$ by some element in G.

We furthermore note that $H^1(\mathbb{C}/\mathbb{R},\mathrm{SL}_n(\mathbb{C})) = H^1(\mathbb{C}/\mathbb{R},\mathrm{Sp}_n(\mathbb{C})) = \{1\}$, see e.g. [32, Chap. X]. Therefore, Condition (2.4.1) holds if $\mathbf{G}$ is a product of factors SL_n or Sp_n or of factors whose group of real points is isomorphic to a complex Lie group viewed as a real Lie group. Note however that $H^1(\mathbb{C}/\mathbb{R},\mathrm{SO}_{p,q}(\mathbb{C}))$ is not trivial; it is in bijective correspondence with the set of non degenerate real quadratic forms that are equivalent to the standard quadratic form of signature (p,q) over $\mathbb{C}$. We emphasize that, in the introduction, we have assumed Condition (2.4.1) to hold.

(2)Note that $m(\pi,\widetilde{\pi},\Gamma)\operatorname{trace}\widetilde{\pi}(f)$ does not depend on the chosen particular extension $\widetilde{\pi}$ but only on π.

2.5. — Under assumption (2.4.1), the map

$$H^1(\sigma, \Gamma) \longrightarrow H^1(\sigma, G)$$

necessarily has trivial image. In other words: if δ represents a class in $H^1(\sigma, \Gamma)$ then δ is conjugate to σ by some element of G. In particular, we have

$$\mathrm{vol}(\Gamma^\delta \backslash G^\delta) = \mathrm{vol}(\Gamma^\sigma \backslash G^\sigma) \quad \text{and} \quad \int_{G^\delta \backslash G} f(x^{-1}\delta x)\, d\dot{x} = \int_{G^\sigma \backslash G} f(x^{-1}\sigma x)\, d\dot{x}.$$

We may therefore write the geometric side of the twisted trace formula as:

$$(2.5.1) \quad |H^1(\sigma, \Gamma)| \,\mathrm{vol}(\Gamma^\sigma \backslash G^\sigma) \int_{G^\sigma \backslash G} f(x^{-1}\sigma x)\, d\dot{x} + \sum_{\substack{\{\delta\} \\ \delta \notin Z^1(\sigma,\Gamma)}} \mathrm{vol}(\Gamma^\delta \backslash G^\delta) \int_{G^\delta \backslash G} f(x^{-1}\delta x)\, d\dot{x}.$$

2.6. Finite dimensional representations of $\widetilde{G}$. — Note that the complexification of G may be identified with the complex points of $\mathrm{Res}_{\mathbb{E}/\mathbb{R}}\, \mathbf{G}$, i.e., $\mathbf{G}(\mathbb{C})^p$. Every complex finite dimensional σ-stable irreducible representation $(\widetilde{\rho}, F)$ of $\widetilde{G}$ can therefore be realized in a space $F = F_0^{\otimes p}$, where (ρ_0, F_0) is an irreducible complex linear representation of $\mathbf{G}(\mathbb{C})$. The action of G is defined by the tensor product action $\rho_0^{\otimes p}$ if $\mathbb{E} = \mathbb{R}^p$ and by $\otimes_{i=1}^{p/2}(\rho_0 \otimes \overline{\rho}_0)$, where $\overline{\rho}_0$ is obtained by composing the complex conjugation in $\mathbf{G}(\mathbb{C})$ by ρ_0, if $\mathbb{E} = \mathbb{C}^{p/2}$. In both cases, we choose the action of σ on $F = F_0^{\otimes p}$ to be the cyclic permutation $A : x_1 \otimes \cdots \otimes x_p \mapsto x_p \otimes x_1 \otimes \cdots \otimes x_{p-1}$. Note that

$$\mathrm{trace}(\sigma \mid F) = \dim F_0.$$

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Say that $(\widetilde{\rho}, F)$ is *strongly twisted acyclic* if there exists a positive constant η depending only on F such that: for every irreducible unitary $(\mathfrak{g}, \widetilde{K})$-module V for which

$$\mathrm{trace}(\sigma \mid C^j(\mathfrak{g}(\mathbb{C}), K, V \otimes F)) \neq 0$$

for some $j \leqslant \dim \mathbf{G}(\mathbb{C})$, the inequality

$$\Lambda_F - \Lambda_V \geqslant \eta$$

is satisfied. Here, Λ_F, resp. Λ_V, is the scalar by which the Casimir acts on F, resp. V.

Write ν for the highest weight of F_0. The following lemma can be proven analogously to [3, Lem. 4.1].

2.7. Lemma. — *Suppose that ν is not preserved by the Cartan involution θ. Then $\widetilde{\rho}$ is strongly twisted acyclic.*

3. Lefschetz number and twisted analytic torsion

Let G, σ and Γ be as in Section 2.3 and let $(\widetilde{\rho}, F)$ be a complex finite dimensional σ-stable irreducible representation of $\widetilde{G}$. We denote by $\mathfrak{g}$ the Lie algebra of G.

3.1. Twisted $(\mathfrak{g}(\mathbb{C}), K)$-cohomology and Lefschetz number. — We can define an action of σ on each cohomology group $H^i(\Gamma\backslash X, F)$ and thus define a *Lefschetz number*

$$\mathrm{Lef}(\sigma, \Gamma, F) = \sum_i (-1)^i \operatorname{trace}(\sigma \mid H^i(\Gamma\backslash X, F)).$$

If V is a $(\mathfrak{g}, \widetilde{K})$-module, we have a natural action of σ on the space of $(\mathfrak{g}, K)$-cochains $C^\bullet(\mathfrak{g}(\mathbb{C}), K, V)$ which induces an action on the quotient $H^\bullet(\mathfrak{g}(\mathbb{C}), K, V)$. We denote by

$$\operatorname{trace}(\sigma \mid H^\bullet(\mathfrak{g}, K, V))$$

the trace of the corresponding operator. We then define the *Lefschetz number* of V by

$$\mathrm{Lef}(\sigma, V) = \sum_i (-1)^i \operatorname{trace}(\sigma \mid H^i(\mathfrak{g}, K, V)).$$

If F is a finite dimensional representation of $\widetilde{G}$ then $F \otimes V$ is still a $(\mathfrak{g}, \widetilde{K})$-module; we denote by $\mathrm{Lef}(\sigma, F, V)$ its Lefschetz number.

Labesse [19, §7] proves that there exists a compactly supported function $L_\rho \in C_c^\infty(\widetilde{G})$ such that for every essential admissible representation $(\widetilde{\pi}, V)$ of $\widetilde{G}$ one has

$$\mathrm{Lef}(\sigma, F, V) = \operatorname{trace} \widetilde{\pi}(L_\rho).$$

The function L_ρ is called the *Lefschetz function* for σ and $(\widetilde{\rho}, F)$.

We then have:

$$\begin{aligned} \operatorname{trace} \widetilde{R}_\Gamma(L_\rho) &= \sum_i (-1)^i \operatorname{trace}(\sigma \mid H^i(\Gamma\backslash X, F)) \\ &= \mathrm{Lef}(\sigma, \Gamma, F). \end{aligned} \tag{3.1.1}$$

3.2. Twisted heat kernels. — Let

$$H_t^{\rho,i} \in \big[C^\infty(G) \otimes \mathrm{End}(\wedge^i(\mathfrak{g}/\mathfrak{k})^* \otimes F)\big]^{K\times K}$$

be the heat kernel for L^2-forms of degree i with values in the bundle associated to (ρ, F). Note that we have a natural action of σ on $\wedge^i(\mathfrak{g}/\mathfrak{k})^* \otimes F$; we denote by A_σ the corresponding linear operator and let

$$h_t^{\rho,i,\sigma} : x \rtimes \sigma \longmapsto \operatorname{trace}(H_t^{\rho,i}(x) \circ A_\sigma).$$

Eventually we shall apply the twisted trace formula to $h_t^{\rho,i,\sigma}$. The heat kernel $H_t^{\rho,i}$ is not compactly supported. However, it follows from [2, Prop. 2.4] that it belongs to all Harish-Chandra Schwartz spaces $\mathscr{C}^q \otimes \mathrm{End}(\wedge^i(\mathfrak{g}/\mathfrak{k})^* \otimes F)$, $q > 0$. This is enough to ensure absolute convergence of both sides of the twisted trace formula.

3.3. Lemma. — *Let $\widetilde{\pi}$ be an essential admissible representation of $\widetilde{G}$ and let V be its associated $(\mathfrak{g}, \widetilde{K})$-module. We have:*

$$\operatorname{trace} \widetilde{\pi}(h_t^{\rho,i,\sigma}) = e^{t(\Lambda_V - \Lambda_F)} \operatorname{trace}(\sigma | [\wedge^i(\mathfrak{g}/\mathfrak{k})^* \otimes F \otimes V]^K).$$

Proof. — It follows from the $K\times K$ equivariance of $H_t^{\rho,i}$ and Kuga's Lemma that, relative to the splitting

$$\wedge^i(\mathfrak{g}/\mathfrak{k})^*\otimes F\otimes V=[\wedge^i(\mathfrak{g}/\mathfrak{k})^*\otimes F\otimes V]^K\oplus\big([\wedge^i(\mathfrak{g}/\mathfrak{k})^*\otimes F\otimes V]^K\big)^{\perp},$$

we have:

$$\pi(H_t^{\rho,i})=\begin{pmatrix}e^{t(\Lambda_V-\Lambda_F)}\mathrm{Id} & 0\\ 0 & 0\end{pmatrix}.$$

We furthermore note that this decomposition is σ-invariant since K is σ-stable. We conclude that we have:

$$\widetilde{\pi}(H_t^{\rho,i}):=\int_G(\pi(g)\circ A)\otimes(H_t^{\rho,i}(g)\circ A_\sigma)\,dg=\begin{pmatrix}e^{t(\Lambda_V-\Lambda_F)}A_\sigma\otimes A & 0\\ 0 & 0\end{pmatrix}.$$

Here, π is the restriction of $\widetilde{\pi}$ to G and A is the intertwining operator between π and $\pi\circ\sigma$ that determines $\widetilde{\pi}$.

Now let $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{e_j\}_{j=1,\dots,m}$ be orthonormal bases of V and $\wedge^i(\mathfrak{g}/k)^*\otimes F$, respectively. Then we have:

$$\begin{aligned}\operatorname{trace}\widetilde{\pi}(H_t^{\rho,i})&=\sum_{n=1}^{\infty}\sum_{j=1}^{m}\langle\widetilde{\pi}(H_t^{\rho,i})(\xi_n\otimes e_j),(\xi_n\otimes e_j)\rangle\\&=\sum_{n=1}^{\infty}\sum_{j=1}^{m}\int_G\langle(\pi(g)\circ A)\xi_n,\xi_n\rangle\langle(H_t^{\rho,i}(g)\circ A_\sigma)e_j,e_j\rangle\,dg\\&=\sum_{n=1}^{\infty}\int_G\langle(\pi(g)\circ A)\xi_n,\xi_n\rangle h_t^{\rho,i,\sigma}(g\rtimes\sigma)\,dg\\&=\operatorname{trace}\widetilde{\pi}(h_t^{\rho,i,\sigma}).\end{aligned}$$

The lemma follows. □

Denoting by $H_t^0\in\big[C^\infty(G)\otimes\mathrm{End}(\wedge^0(\mathfrak{g}/\mathfrak{k})^*)\big]^{K\times K}$ the heat kernel for L^2-functions on X, the following proposition follows from [27, Prop. 5.3] and the definition of strong twisted acyclicity.

3.4. Proposition. — *Assume that $(\widetilde{\rho},F)$ is strongly twisted acyclic. Then there exist positive constants η and C such that for every $x\in G$, $t\in(0,+\infty)$ and $i\in\{0,\dots,\dim X\}$, one has:*

$$|h_t^{\rho,i,\sigma}(x\rtimes\sigma)|\leqslant Ce^{-\eta t}H_t^0(x).$$

We define the kernel $k_t^{\rho,\sigma}$ by

$$k_t^{\rho,\sigma}(g)=\sum_i(-1)^i i h_t^{\rho,i,\sigma}(g);$$

it defines a function in $\mathscr{C}^q(\widetilde{G})$, for all $q>0$.

3.5. Twisted analytic torsion. — The *twisted analytic torsion* $T^{\sigma}_{\Gamma\backslash X}(\rho)$ is then defined by

(3.5.1) $$\log T^{\sigma}_{\Gamma\backslash X}(\rho) = \frac{1}{2}\frac{d}{ds}\Big|_{s=0}\bigg(\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\Big[\operatorname{trace}\widetilde{R}_{\Gamma}(k_t^{\rho,\sigma}) - \sum_i (-1)^i i\cdot \operatorname{trace}(\sigma \mid H^i(\Gamma\backslash X, F))\Big]dt\bigg).$$

Note that if $(\widetilde{\rho}, F)$ is strongly twisted acyclic each $\operatorname{trace}(\sigma \mid H^i(\Gamma\backslash X, F))$ is trivial. From now on we will assume that $(\widetilde{\rho}, F)$ is strongly twisted acyclic. In particular, we have:

$$\log T^{\sigma}_{\Gamma\backslash X}(\rho) = \frac{1}{2}\frac{d}{ds}\Big|_{s=0}\bigg(\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\operatorname{trace}\widetilde{R}_{\Gamma}(k_t^{\rho,\sigma})dt\bigg).$$

3.6. Twisted $(\mathfrak{g}, K)$-torsion. — If V is a $(\mathfrak{g}, \widetilde{K})$-module and F is a finite dimensional representation of $\widetilde{G}$ then $F\otimes V$ is still a $(\mathfrak{g}, \widetilde{K})$-module; we define the *twisted* $(\mathfrak{g}, K)$-*torsion* of $F\otimes V$ by

$$\begin{aligned}\operatorname{Lef}'(\sigma, F, V) &= \sum_i (-1)^i i \operatorname{trace}(\sigma \mid C^i(\mathfrak{g}, K, F\otimes V))\\ &= \sum_i (-1)^i i \operatorname{trace}(\sigma \mid [\wedge^i(\mathfrak{g}/\mathfrak{k})^*\otimes F\otimes V]^K).\end{aligned}$$

Remark. — We should explain the notation Lef'. Given a group G and a G-vector space V, we denote by $\det[1-V]$ the virtual G-representation (that is to say, element of K_0 of the category of G-representations) defined by the alternating sum $\sum_i(-1)^i[\wedge^i V]$ of exterior powers. This is multiplicative in an evident sense:

(3.6.1) $$\det[1-V\oplus W] = \det[1-V]\otimes\det[1-W].$$

Now given $g\in G$, the derivative $\frac{d}{dt}\big|_{t=1}\det(t1-g)$ is equal to the character of g acting on $\sum_i(-1)^i i\wedge^i V$. We therefore define $\det'[1-V] = \sum_i(-1)^i i\wedge^i V$.

Considering the virtual $\widetilde{K}$-representation $\det'[1-(\mathfrak{g}/\mathfrak{k})^*]$ we have:

(3.6.2) $$\operatorname{Lef}'(\sigma, F, V) = \operatorname{trace}\Big(\sigma \mid [\det'[1-(\mathfrak{g}/\mathfrak{k})^*]\otimes F\otimes V]^K\Big).$$

This explains our notation Lef' for the twisted $(\mathfrak{g}, K)$-torsion.

For future reference we note that we have:

(3.6.3) $$\det'[1-V\oplus W] = \det'[1-V]\otimes\det[1-W]\oplus\det[1-V]\otimes\det'[1-W].$$

We also note that Labesse's proof of the existence of L_ρ can easily be modified to get a function $L'_\rho\in C_c^{\infty}(\widetilde{G})$ such that for every essential admissible representation $(\widetilde{\pi}, V)$ of $\widetilde{G}$ one has

$$\operatorname{Lef}'(\sigma, F, V) = \operatorname{trace}\widetilde{\pi}(L'_\rho).$$

3.7. — It follows from Lemma 3.3 that the spectral side of the twisted trace formula evaluated in $k_t^{\rho,\sigma}$ is

$$\operatorname{trace} \widetilde{R}_\Gamma(k_t^{\rho,\sigma}) = \sum_{\pi\in\Pi(\widetilde{G})} m(\pi,\widetilde{\pi},\Gamma)\operatorname{Lef}'(\sigma,F,V_\pi)\, e^{t(\Lambda_\pi-\Lambda_F)}, \tag{3.7.1}$$

where V_π is the $(\mathfrak{g},\widetilde{K})$-module associated to the extension $\widetilde{\pi}$.

4. L^2-Lefschetz number, L^2-torsion and limit formulas

Let G, σ and Γ be as in Section 2.3 and let $(\widetilde{\rho},F)$ be as in the preceding sections. Suppose that $(\widetilde{\rho},F)$ is strongly twisted acyclic. Let $f\in C_c^\infty(\widetilde{G})$.

4.1. — Given $g\in G$ we define $r(g)=\operatorname{dist}(gK,eK)$ with respect to the Riemannian symmetric distance of $X=G/K$. We extend r to $G\rtimes\langle\sigma\rangle$ by setting $r(g\rtimes\sigma)=r(g)$. Note that $r(gg')\leqslant r(g)+r(g')$.

Now, given $x\in G$ we set $\ell(x)=\inf\{r(gxg^{-1}) : g\in G\}$; it only depends on the conjugacy class of x in G. Recall that the *injectivity radius of* Γ,

$$r_\Gamma=\frac{1}{2}\inf\{\ell(\gamma) : \gamma\in\Gamma-\{e\}\},$$

is strictly positive. If $\delta\in\widetilde{\Gamma}$ with $\delta\notin Z^1(\sigma,\Gamma)$, then $\delta^p\in\Gamma-\{1\}$ and therefore $\ell(\delta^p)\geqslant 2r_\Gamma$. In particular, for any $x\in G$ we have:

$$2r_\Gamma\leqslant\ell(\delta^p)\leqslant r(x\delta^p x^{-1})=r((x\delta x^{-1})\cdots(x\delta x^{-1}))\leqslant p\cdot r(x\delta x^{-1}). \tag{4.1.1}$$

4.2. Lemma. — *There exist constants $c_1,c_2>0$, depending only on G, such that for any $x\in G$, we have:*

$$N(x;R):=\bigl|\bigl\{\delta\in\widetilde{\Gamma} : \delta\notin Z^1(\sigma,\Gamma)\quad\textit{and}\quad r(x\delta x^{-1})\leqslant R\bigr\}\bigr|\leqslant c_1p^d r_\Gamma^{-d}e^{c_2R},$$

where d is the dimension of X.

Proof. — It follows from (4.1.1) that it suffices to prove the lemma for $R\geqslant 2r_\Gamma/p$. Set $\varepsilon:=r_\Gamma/p$. By definition of $r(x\delta x^{-1})$ we have $B(\gamma(\sigma(x)),\varepsilon)\subset B(x,R+\varepsilon)$, for all $\delta=\gamma\rtimes\sigma\in\widetilde{\Gamma}$ with $\delta\notin Z^1(\sigma,\Gamma)$ and $r(x\delta x^{-1})\leqslant R$. Now, since $\varepsilon<r_\Gamma$, the balls $B(\gamma(\sigma(x)),\varepsilon)$, $\gamma\in\Gamma$, are all disjoint of the same volume. We conclude that

$$N(x;R)\cdot\operatorname{vol}B(\sigma(x),\varepsilon)\leqslant\operatorname{vol}B(x,R+\varepsilon)\leqslant\operatorname{vol}B(x,2R).$$

We conclude using standard estimates on volumes of balls (see e.g. [1, Lem. 7.21] for more details). □

4.3. — Now, let $\{\Gamma_n\}$ be a normal chain with $\bigcap_n\Gamma_n=\{1\}$ or more generally a family of finite index normal subgroups such that for every $1\neq\gamma\in\Gamma$, the set $\{n:\gamma\in\Gamma_n\}$ is finite. We have:

4.4. Lemma. — *Let $\delta\notin Z^1(\sigma,\Gamma)$. There exists a constant n_δ such that for every $n\geqslant n_\delta$,*

$$|\{\gamma\in\Gamma_n\backslash\Gamma : \gamma\delta\gamma^{-1}\in\widetilde{\Gamma}_n\}|=0.$$

Proof. — Recall that for $h \in \Gamma$, we define

$$\mathrm{Norm}(h) := h\sigma(h)\cdots\sigma^{p-1}(h).$$

Writing $\delta = h \rtimes \sigma$, the condition $\delta \notin Z^1(\sigma,\Gamma)$ is equivalent to $\mathrm{Norm}(h) \neq 1$. Suppose that $\gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n$ for some $\gamma \in \Gamma_n\backslash\Gamma$. Equivalently, $\epsilon := \gamma h\sigma(\gamma^{-1}) \in \Gamma_n$. Therefore,

$$\mathrm{Norm}(\epsilon) = \gamma\,\mathrm{Norm}(h)\gamma^{-1} \in \Gamma_n.$$

Since Γ_n is normal in Γ, this implies that $\mathrm{Norm}(h) \in \Gamma_n$. Since $\mathrm{Norm}(h) \neq 1$, it follows that $\mathrm{Norm}(h) \in \Gamma_n$ for at most finitely many n. Therefore,

$$\left|\left\{\gamma \in \Gamma_n\backslash\Gamma \,:\, \gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n\right\}\right| = 0$$

for all but finitely many n. □

We will also need the following:

4.5. Lemma. — *There is a uniform upper bound*

$$\frac{|\{\gamma \in \Gamma_n\backslash\Gamma : \gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n\}|}{[\Gamma^\sigma : \Gamma_n^\sigma]\cdot|H^1(\sigma,\Gamma_n)|} \leqslant 1.$$

Proof. — Write $\delta = h \rtimes \sigma$. Let $\gamma_0 \in \Gamma_n\backslash\Gamma$ satisfy $\gamma_0\sigma\gamma_0^{-1} \in \widetilde{\Gamma}_n$. Equivalently, we have $\gamma_0 h\sigma(\gamma_0^{-1}) \in \Gamma_n$. Then we have

$$h\sigma(\gamma_0^{-1})\gamma_0 \in \gamma_0^{-1}\Gamma_n\gamma_0 = \Gamma_n \iff h\Gamma_n = \gamma_0^{-1}\sigma(\gamma_0)\Gamma_n.$$

Suppose $\gamma \in \Gamma_n\backslash\Gamma$ is another element satisfying $\gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n$. Then similarly $h\Gamma_n = \gamma^{-1}\sigma(\gamma)$, from which it follows that $\sigma(\gamma\gamma_0^{-1}) = \gamma\gamma_0^{-1}$. Conversely, if $\gamma = g\gamma_0$ for some $g \in (\Gamma_n\backslash\Gamma)^\sigma$, then $\gamma h\sigma(\gamma^{-1}) \in \Gamma_n$. Therefore,

$$|\{\gamma \in \Gamma_n\backslash\Gamma : \gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n\}| = |\,(\Gamma_n\backslash\Gamma)^\sigma\,|.$$

To estimate the size of $(\Gamma_n\backslash\Gamma)^\sigma$, we use the long exact sequence of pointed sets

$$1 \longrightarrow \Gamma_n^\sigma \longrightarrow \Gamma^\sigma \longrightarrow (\Gamma_n\backslash\Gamma)^\sigma \xrightarrow{\ d\ } H^1(\sigma,\Gamma_n) \longrightarrow H^1(\sigma,\Gamma).$$

The fibers of the connecting map d are orbits of Γ^σ acting on $(\Gamma_n\backslash\Gamma)^\sigma$ [32, I, §5.4, Cor. 1]. Because Γ_n is normal in Γ, the action of Γ^σ on $(\Gamma_n\backslash\Gamma)^\sigma$ factors through $\Gamma_n^\sigma\backslash\Gamma^\sigma$. Therefore, all fibers of d have size at most $\#(\Gamma_n^\sigma\backslash\Gamma^\sigma)$. Also,

$$\begin{aligned}\#\mathrm{image}(d) &= \#\ker[H^1(\sigma,\Gamma_n) \longrightarrow H^1(\sigma,\Gamma)]\\ &\leqslant \#H^1(\sigma,\Gamma_n).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{|(\Gamma_n\backslash\Gamma)^\sigma|}{[\Gamma^\sigma : \Gamma_n^\sigma]\cdot|H^1(\sigma,\Gamma_n)|} &\leqslant \frac{\#\mathrm{image}(d)\cdot\#\text{largest fiber}}{[\Gamma^\sigma : \Gamma_n^\sigma]\cdot|H^1(\sigma,\Gamma_n)|}\\ &\leqslant \frac{\#H^1(\sigma,\Gamma_n)\cdot\#(\Gamma_n^\sigma\backslash\Gamma^\sigma)}{[\Gamma^\sigma : \Gamma_n^\sigma]\cdot|H^1(\sigma,\Gamma_n)|}\\ &\leqslant 1.\end{aligned}$$

□

We next give a typical example of a sequence of σ-stable normal subgroups $\Gamma_n \lhd \Gamma$ satisfying the hypotheses of Section 4.3.

Example. — Let $\mathbf{G}/O_{F,S}$ be a semisimple group, where $O_{F,S}$ denotes the ring of S-integers in a number field F. Let E/F be a cyclic Galois extension with $\mathrm{Gal}(E/F) = \langle\sigma\rangle$. We fix an integral structure on $\mathbf{G}$ so we may speak of $\mathbf{G}(O_F)$ and $\mathbf{G}(O_E)$.

Fix a finite index, σ-stable subgroup $\Gamma \subset \mathbf{G}(O_E)$. Let $\mathfrak{n} \subset O_F$ be an ideal. Let

$$\Gamma_{\mathfrak{n}} := \{\gamma \in \Gamma : \gamma \equiv 1 \bmod \mathfrak{n}O_E\}. \tag{4.5.1}$$

Every $\Gamma_{\mathfrak{n}} \subset G = \mathbf{G}(E_{\mathbb{R}})$ is a Galois-stable lattice. If $\mathfrak{n}$ varies through any sequence $\mathscr{F}$ of ideals satisfying $\mathrm{Norm}(\mathfrak{n}) \to \infty$, then $\{\Gamma_{\mathfrak{n}}\}_{n\in\mathscr{F}}$ satisfies the hypothesis of Section 4.3.

4.6. Proposition. — *Let $f \in C_c^\infty(\widetilde{G})$ and let $\{\Gamma_n\}$ be a sequence of finite index σ-stable subgroups of Γ that satisfies the hypothesis of Section 4.3. Then*

$$\frac{\operatorname{trace} \widetilde{R}_{\Gamma_n}(f)}{|H^1(\sigma,\Gamma_n)| \operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma)} \longrightarrow \int_{G^\sigma\backslash G} f(x^{-1}\sigma x)\, d\dot{x}.$$

Proof. — It follows from (2.5.1) that we have:

$$\begin{aligned}\operatorname{trace} \widetilde{R}_{\Gamma_n}(f) &= |H^1(\sigma,\Gamma_n)| \operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma) \int_{G^\sigma\backslash G} f(x^{-1}\sigma x)\, d\dot{x} \\ &\qquad + \sum_{\substack{\{\delta\}_{\Gamma_n} \\ \delta\notin Z^1(\sigma,\Gamma_n)}} \operatorname{vol}(\Gamma_n^\delta\backslash G^\delta) \int_{G^\delta\backslash G} f(x^{-1}\delta x)\, d\dot{x} \\ &= |H^1(\sigma,\Gamma_n)| \operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma) \int_{G^\sigma\backslash G} f(x^{-1}\sigma x)\, d\dot{x} \\ &\qquad + \sum_{\substack{\{\delta\}_{\Gamma} \\ \delta\notin Z^1(\sigma,\Gamma)}} c_{\Gamma_n}(\delta) \operatorname{vol}(\Gamma^\delta\backslash G^\delta) \int_{G^\delta\backslash G} f(x^{-1}\delta x)\, d\dot{x},\end{aligned}$$

where

$$c_{\Gamma_n}(\delta) = \big|\big\{\gamma \in \Gamma_n\backslash\Gamma \,:\, \gamma\delta\gamma^{-1} \in \widetilde{\Gamma}_n\big\}\big|.$$

Since f is compactly supported, the last sum above is finite: choosing $R > 0$ so that the support of f is contained in

$$B_R = \{g \rtimes \sigma \in \widetilde{G} \,:\, r(g) \leqslant R\},$$

we may restrict the sum on the right side of the above equation to the δ's that are contained in B_R. It follows from Lemma 4.2 that the corresponding sum is finite. We conclude the proof of Proposition 4.6 by applying Lemma 4.4 to the finitely many δ's appearing in this finite sum. □

Remark. — In the untwisted case, the condition

$$\frac{|\{\gamma \in \Gamma_n\backslash\Gamma \,:\, \gamma\delta\gamma^{-1} \in \Gamma_n\}|}{[\Gamma:\Gamma_n]} \longrightarrow 0$$

is equivalent to the BS-convergence of the *compact* quotients $\Gamma_n\backslash X$ towards the symmetric space X. See [1].

4.7. L^2-Lefschetz number. — Proposition 4.6 motivates the following definition of the *L^2-Lefschetz number* associated to the triple (G,σ,ρ):

$$\mathrm{Lef}^{(2)}(\sigma,X,F)=\int_{G^\sigma\backslash G}L_\rho(x^{-1}\sigma x)\,d\dot{x}.$$

4.8. Corollary. — *Let $\{\Gamma_n\}$ be a normal chain of finite index σ-stable subgroups of Γ that satisfies the hypothesis of Section 4.3. Then we have:*

$$\frac{\mathrm{Lef}(\sigma,\Gamma_n,F)}{|H^1(\sigma,\Gamma_n)|\,\mathrm{vol}(\Gamma_n^\sigma\backslash G^\sigma)}\longrightarrow \mathrm{Lef}^{(2)}(\sigma,X,F).$$

Proof. — Apply Proposition 4.6 to the Lefschetz function L_ρ. □

4.9. Twisted L^2-torsion

Analogously we define the *twisted L^2-torsion* $T^{(2)\sigma}_{\Gamma\backslash X}(\rho)\in\mathbb{R}^+$ by

$$\log T^{(2)\sigma}_{\Gamma\backslash X}(\rho)=|H^1(\sigma,\Gamma)|\,\mathrm{vol}(\Gamma^\sigma\backslash G^\sigma)t^{(2)\sigma}_X(\rho), \tag{4.9.1}$$

where $t^{(2)\sigma}_X(\rho)$ — which depends only on the symmetric space X, the involution σ, and the finite dimensional representation ρ — is defined by

$$t^{(2)\sigma}_X(\rho)=\frac12\frac{d}{ds}\Big|_{s=0}\bigg(\frac{1}{\Gamma(s)}\int_0^\infty\int_{G^\sigma\backslash G}k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\,t^{s-1}dt\bigg). \tag{4.9.2}$$

Note that $k_t^{\rho,\sigma}$ is not compactly supported and that we have to prove that the RHS of (4.9.2) is indeed well-defined. Recall however that $k_t^{\rho,\sigma}$ belongs to $\mathscr{C}^q(\widetilde{G})$. Lemma 4.2 therefore implies that the series

$$\sum_{\delta\in\widetilde{\Gamma}}k_t^{\rho,\sigma}(x^{-1}\delta x)$$

converges absolutely and locally uniformly. This implies that the integral of this series along a (compact) fundamental domain D for the action of Γ on G is absolutely convergent. Restricting the sum to the δ's that belong to the (twisted) Γ-conjugacy class of σ we conclude in particular that, for every positive t, the integral

$$\int_{G^\sigma\backslash G}k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}=\frac{1}{\mathrm{vol}(\Gamma^\sigma\backslash G^\sigma)}\int_D\sum_{\delta\in\{\sigma\}}k_t^{\rho,\sigma}(x^{-1}\delta x)\,dx$$

is absolutely convergent. We postpone the proof of the fact that (4.9.2) is indeed well-defined until sections 6 and 7 where we will explicitly compute $t^{(2)\sigma}_X(\rho)$. In the course of the computations we will also prove the following lemma.

4.10. Lemma. — *There exist constants $C,c>0$ such that*

$$\bigg|\int_{G^\sigma\backslash G}k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\bigg|\leqslant Ce^{-ct},\quad t\geqslant 1. \tag{4.10.1}$$

Taking this lemma for granted, we conclude this section by the proof of the following 'limit multiplicity theorem'.

4.11. Theorem. — *Assume that $(\widetilde{\rho}, F)$ is strongly twisted acyclic. Let $\{\Gamma_n\}$ be a sequence of finite index σ-stable subgroups of Γ that satisfies the hypothesis of Section 4.3. Then*

$$\frac{\log T^{\sigma}_{\Gamma_n\backslash X}(\rho)}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} \longrightarrow t_X^{(2)\sigma}(\rho).$$

Proof. — Since $k_t^{\rho,\sigma} \in \mathscr{C}^q(\widetilde{G})$, for all $q>0$, we still have:

$$\begin{aligned}\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma}) = |H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma}) \int_{G^{\sigma}\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\\ + \sum_{\substack{\{\delta\}_{\Gamma}\\ \delta\notin Z^1(\sigma,\Gamma)}} c_{\Gamma_n}(\delta)\operatorname{vol}(\Gamma^{\delta}\backslash G^{\delta}) \int_{G^{\delta}\backslash G} k_t^{\rho,\sigma}(x^{-1}\delta x)\,d\dot{x}.\end{aligned}$$

Note that at this point it is not clear that the sum on the right (absolutely) converges. This is however indeed the case: it first follows from (4.1.1) that if $\delta \notin Z^1(\sigma,\Gamma)$ and $x\in G$ we have $r(x\delta x^{-1}) \geqslant 2r_{\Gamma}/p$. Now, recall from [3, Lem. 3.8] or [27, Prop. 3.1 and (3.14)] that we have, for $t\in(0,1]$,

$$|k_t^{\rho,\sigma}(x^{-1}\delta x)| \leqslant Ct^{-d}\exp\Bigl(-c\frac{r(x\delta x^{-1})^2}{t}\Bigr) \leqslant Ce^{-c'/t}\exp\bigl(-c''r(x\delta x^{-1})^2\bigr). \tag{4.11.1}$$

(Here, c' depends on r_{Γ}.) From this and Lemma 4.2, it follows that the geometric side of the trace formula evaluated in $k_t^{\rho,\sigma}$ indeed absolutely converges. Moreover, it follows from (4.11.1) together with the uniform bound of Lemma 4.5 that we have:

$$\frac{\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} = \int_{G^{\sigma}\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x} + O(e^{-c'/t}) \quad (0<t\leqslant 1), \tag{4.11.2}$$

where the implied constant in the $O(e^{-c'/t})$ is uniform in n. We conclude that

$$\int_0^1 t^{s-1}\biggl(\frac{\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} - \int_{G^{\sigma}\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\biggr)dt$$

is holomorphic is s in a half-plane containing 0, so that

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\biggl(\frac{\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} - \int_{G^{\sigma}\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\biggr)dt\\ = \int_0^{+\infty}\biggl(\frac{\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} - \int_{G^{\sigma}\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\biggr)\frac{dt}{t}.\end{aligned}$$

Now it follows from Proposition 3.4 that there exists some positive η such that $|k_t^{\rho,\sigma}(x\rtimes\sigma)| \ll e^{-\eta t}H_t^0(x)$. In particular, $|k_t^{\rho,\sigma}(x\rtimes\sigma)| \ll e^{-\eta t}H_1^0(x)$ if $t\geqslant 1$ and we have:

$$\frac{|\operatorname{trace} \widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})|}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} \ll e^{-\eta t}\sum_{\{\delta\}_{\Gamma}}\int_{G^{\delta}\backslash G} H_1^0(x^{-\sigma}\delta x)\,d\dot{x}.$$

The sum on the right-hand-side splits into an infinite sum over the δ's not in $Z^1(\sigma,\Gamma)$ and the finite sum of the remaining terms. By (4.11.1) the infinite sum is absolutely convergent and independent of t and n. We use Lemma 4.10 to bound the finite sum. And we get:

$$\frac{|\operatorname{trace}\widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})|}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma)} \ll e^{-\eta t},$$

where the implicit constant does not depend on n. Using Lemma 4.10, we conclude that both

$$\int_1^{+\infty}\frac{\operatorname{trace}\widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma)}\frac{dt}{t} \quad\text{and}\quad \int_1^{+\infty}\int_{G^\sigma\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}\frac{dt}{t}$$

are absolutely convergent uniformly in n.

It follows from Equation (4.11.2) and the last paragraph (the former for $0 < t \leqslant 1$ and the latter for $t \geqslant 1$) that

$$\frac{\operatorname{trace}\widetilde{R}_{\Gamma_n}(k_t^{\rho,\sigma})}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma)} - \int_{G^\sigma\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\,d\dot{x}$$

is bounded by a function of t which does not depend on n and is absolutely integrable on $(0,+\infty)$. From this and Lebesgue's dominated convergence theorem, we conclude that Proposition 4.6 implies Theorem 4.11. □

5. Computations on a product

In this section we compute the L^2-Lefschetz numbers and twisted L^2-torsion and in particular prove Lemma 4.10 in the case where $\mathbb{E}=\mathbb{R}^p$ (the product case).

Here, we suppose that $\mathbb{E}=\mathbb{R}^p$. Then G is the p-fold product of $\mathbf{G}(\mathbb{R})$ and σ cyclically permutes the factors of G. We will abusively denote by G^σ the group $\mathbf{G}(\mathbb{R})$. Let (ρ_0, F_0) be an irreducible complex linear representation of $\mathbf{G}(\mathbb{C})$. We denote by $(\widetilde{\rho}, F)$ the corresponding complex finite dimensional σ-stable irreducible representation of $\widetilde{G}$. Recall that $F=F_0^{\otimes p}$, that G acts by the tensor product representation $\rho_0^{\otimes p}$ and that σ acts by the cyclic permutation $A : x_1\otimes\cdots\otimes x_p \mapsto x_p\otimes x_1\otimes\cdots\otimes x_{p-1}$. We finally let X and X^σ be the symmetric spaces corresponding to G and G^σ respectively, so $X=(X^\sigma)^p$.

5.1. Heat kernels of a product. — The heat kernels $H_t^{\rho,j}$ decompose as

$$H_t^{\rho,j}(g_1,\cdots,g_p)=\sum_{a_1+\cdots+a_p=j} H_t^{\rho_0,a_1}(g_1)\otimes\cdots\otimes H_t^{\rho_0,a_p}(g_p). \tag{5.1.1}$$

Now the twisted orbital integral of $H_t^{\rho,j}$ associated to the class of the identity element is given by

$$\left(\int_{G^\sigma\backslash G} H_t^{\rho,j}(g^{-\sigma}g)\,dg\right)\circ A_\sigma.$$

But because $H_t^{\rho,j}(g^{-\sigma}g)$ preserves all of the summands in the decomposition of (5.1.1) and σ maps the $(a_1,\dots,a_p)$-summand to the $(a_p,a_1,\dots,a_{p-1})$-summand, only those summands for which $j=pa$ and $a_1=\cdots=a_p=a$ can contribute to the trace of the above twisted orbital integral. Furthermore, by a computation identical to that done for scalar-valued functions in [21, §8], we see that

$$\left(\int_{G^\sigma\backslash G} H_t^{\rho_0,a}(g_p^{-1}g_1)\otimes\cdots\otimes H_t^{\rho_0,a}(g_{p-1}^{-1}g_p)\,dg\right)\circ A_\sigma = (-1)^{a^2(p-1)} H_t^{\rho_0,a} *\cdots* H_t^{\rho_0,a}(e).$$

This implies that

$$\begin{aligned}\int_{G^\sigma\backslash G} h_t^{\rho,pa}(x^{-1}\sigma x)\,d\dot{x} &= \operatorname{trace}\left[\left(\int_{G^\sigma\backslash G} H_t^{\rho_0,a}(g_p^{-1}g_1)\otimes\cdots\otimes H_t^{\rho_0,a}(g_{p-1}^{-1}g_p)\,d\dot{g}\right)\circ A_\sigma\right]\\ &= (-1)^{a^2(p-1)}\operatorname{trace}\left(H_t^{\rho_0,a} *\cdots * H_t^{\rho_0,a}(e)\right)\\ &= (-1)^{a^2(p-1)}\operatorname{trace}\left(H_{pt}^{\rho_0,a}(e)\right)\\ &= (-1)^{a^2(p-1)} h_{pt}^{\rho_0,a}(e).\end{aligned}$$

Here, $H_{pt}^{\rho_0,a}$ is an *untwisted* heat kernel on X^σ. Lemma 4.10 therefore follows from standard estimates (see e.g. [3]). Moreover, computations of the L^2-Lefschetz number and of the twisted L^2-torsion immediately follow from the above explicit computation.

5.2. Theorem (L^2-Lefschetz number of a product). — *We have:*

$$\operatorname{Lef}^{(2)}(\sigma,X,F) = \begin{cases}(-1)^{\frac{1}{2}\dim X^\sigma}(\dim F_0)\,\dfrac{\chi(X_u^\sigma)}{\operatorname{vol}(X_u^\sigma)} & \text{if } \delta(G^\sigma)=0,\\ 0 & \text{if not.}\end{cases}$$

Here, X_u^σ is the compact dual of X^σ whose metric is normalized such that multiplication by i becomes an isometry $T_{eK^\sigma}(X^\sigma)\cong\mathfrak{p}\to i\mathfrak{p}\cong T_{eK^\sigma}(X_u^\sigma)$.

Proof. — First note[(3)] that

$$\begin{aligned}\operatorname{Lef}^{(2)}(\sigma,X,F) &= \lim_{t\to+\infty}\int_{G^\sigma\backslash G} k_t^\rho(x^{-1}\sigma x)\,d\dot{x}\\ &= \lim_{t\to+\infty}\sum_a(-1)^{pa}\int_{G^\sigma\backslash G} h_t^{\rho,pa}(x^{-1}\sigma x)\,d\dot{x}\\ &= \lim_{t\to+\infty}\sum_a(-1)^a h_{pt}^{\rho_0,a}(e).\end{aligned}$$

The computation then reduces to the untwisted case for which we refer to [28]. □

The computation of the twisted L^2-torsion similarly reduces to the untwisted case:

5.3. Theorem (Twisted L^2-torsion of a product). — *We have:*

$$t_X^{(2)\sigma}(\rho) = p\cdot t_{X^\sigma}^{(2)}(\rho).$$

(3) Beware that ρ is not assumed to be strongly acyclic here!

6. Computations in the case $\mathbb{E} = \mathbb{C}$

In this section we compute the L^2-Lefschetz numbers and twisted L^2-torsion and in particular prove Lemma 4.10 in the case where $\mathbb{E} = \mathbb{C}$. The general case easily reduces to this case and the one of the preceding section.

Throughout this section, $\mathbb{E} = \mathbb{C}$. Then $G = \mathbf{G}(\mathbb{C})$ is the group of complex points, $\sigma : G \to G$ is the real involution given by the complex conjugation and $G^\sigma = \mathbf{G}(\mathbb{R})$. Recall that we fix a choice of Cartan involution θ of G that commutes with σ.

6.1. Irreducible σ-stable tempered representations of G. — Choose θ-stable representatives $\mathfrak{h}_1^0, \ldots, \mathfrak{h}_s^0$ of the $\mathbf{G}(\mathbb{R})$-conjugacy classes of Cartan subalgebras in the Lie algebra $\mathfrak{g}^0$ of $\mathbf{G}(\mathbb{R})$. For each $j \in \{1, \ldots, s\}$ we write $\mathfrak{h}_j^0 = \mathfrak{t}_j \oplus \mathfrak{a}_j$ for the decomposition of $\mathfrak{h}_j^0$ w.r.t. θ, i.e., $\mathfrak{a}_j$ is the split part of $\mathfrak{h}_j^0$ and $\mathfrak{t}_j$ is the compact part of $\mathfrak{h}_j^0$. We denote by $\mathfrak{h}_j$ the complexification of $\mathfrak{h}_j^0$; note that $\mathfrak{a}_j \oplus i\mathfrak{t}_j$ and $\mathfrak{t}_j \oplus i\mathfrak{a}_j$ are resp. the split and compact part of $\mathfrak{h}_j$.

We now fix some j. To ease notation we will omit the j index. Choose a Borel subgroup B of $G = \mathbf{G}(\mathbb{C})$ containing the torus H which corresponds to $\mathfrak{h}_j$. Let A and T be resp. the split and compact tori corresponding to $\mathfrak{a} \oplus i\mathfrak{t}$ and $\mathfrak{t} \oplus i\mathfrak{a}$. Write μ for the differential of a character of T and λ for the differential of a character of A. Note that μ is σ-stable if and only if μ is zero on $i\mathfrak{a}$.

Associated to (μ, λ) is a representation

$$\pi_{\mu,\lambda} = \mathrm{ind}_B^G(\mu \otimes \lambda \otimes 1).$$

6.2. Proposition (Delorme [14]). — *Every irreducible σ-stable tempered representation of G is equivalent to some $\pi_{\mu,\lambda}$ as above (for some j) where μ is zero on $i\mathfrak{a}_j$ and λ is zero on $\mathfrak{t}_j$ and has pure imaginary image.*

Note that if λ is zero on $\mathfrak{t}_j$ we may think of λ as a real linear form $\mathfrak{a} \to \mathbb{C}$. We denote by $I_{\mu,\lambda}$ the underlying $(\mathfrak{g}, K)$-module. It is σ-stable and Delorme [14, §5.3] defines a particular extension to a $(\mathfrak{g}, \widetilde{K})$-module, but we won't follow his convention here (see Convention I below).

6.3. Computations of the Lefschetz numbers. — If $\mathbf{G}(\mathbb{R})$ has no discrete series Delorme [14, Prop. 7] proves that for any admissible $(\mathfrak{g}, \widetilde{K})$-module and any finite dimensional representation $(\widetilde{\rho}, F)$ of $\widetilde{G}$, we have:

$$\mathrm{Lef}(\sigma, F, V) = 0.$$

Even if $\mathbf{G}(\mathbb{R})$ has discrete series, Delorme's proof — see also [30, Lem. 4.2.3] — shows that

$$\mathrm{Lef}(\sigma, F, I_{\mu,\lambda}) = 0$$

unless $\mathfrak{h}^0 = \mathfrak{t}$ is a compact Cartan subalgebra (so that $i\mathfrak{t}$ is the split part of $\mathfrak{h}$). In the latter case $\lambda = 0$ (recall that $I_{\mu,\lambda}$ is assumed to be σ-stable); we will simply denote by I_μ the $(\mathfrak{g}, \widetilde{K})$-module $I_{\mu,0}$. The following proposition — due to Delorme [14, Th. 2][4] — computes the Lefschetz numbers in the remaining cases. Denote by ρ

(4) Note that Delorme considers σ-invariants rather than traces, this introduces a factor $1/2$.

the half-sum of the roots of $\mathfrak{h}(\mathbb{C})$ in B.[5] And recall that ν is a highest weight of F that is dominant with respect to the positive weights defined by B.

6.4. Proposition. — *We have:*

$$\operatorname{Lef}(\sigma, F, I_\mu) = \begin{cases} \pm 2^{\dim \mathfrak{t}} & \textit{if } w\mu = 2(\nu+\rho)_{|\mathfrak{t}} \quad (w \in W), \\ 0 & \textit{otherwise.} \end{cases}$$

Here, W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and the sign depends on the chosen extension of I_μ to a $(\mathfrak{g}, \widetilde{K})$-module.

Convention I. — In the following we will always assume that the extension of a σ-discrete I_μ to a $(\mathfrak{g}, \widetilde{K})$-module is such that that the sign in Proposition 6.4 is positive. (See [30, §4.2.5] for more details.)

6.5. Computations of the twisted $(\mathfrak{g}, K)$-torsion. — Consider an arbitrary irreducible σ-stable tempered representation of G associated to some j and some (μ, λ) as in Proposition 6.2. Let $\mathbf{P}$ be the parabolic subgroup of $\mathbf{G}$ whose Levi subgroup $\mathbf{M} = {}^0\mathbf{M}\mathbf{A}_P$ is the centralizer in $\mathbf{G}$ of $\mathfrak{a}$. We have $B \subset \mathbf{P}(\mathbb{C})$ and we may write $\pi_{\mu,\lambda}$ as the induced representation

$$\pi_{\mu,\lambda} = \operatorname{ind}_{\mathbf{P}(\mathbb{C})}^{\mathbf{G}(\mathbb{C})} (\pi_{\mu,0}^{{}^0\mathbf{M}(\mathbb{C})} \otimes \lambda),$$

where

$$\pi_{\mu,0}^{{}^0\mathbf{M}(\mathbb{C})} = \operatorname{ind}_{B\cap {}^0\mathbf{M}(\mathbb{C})}^{{}^0\mathbf{M}(\mathbb{C})} (\mu_{|\mathfrak{t}} \otimes 0)$$

is a tempered (σ-discrete) representation of ${}^0\mathbf{M}(\mathbb{C})$ and we think of λ — seen as real linear form $\mathfrak{a} \to \mathbb{C}$ — as (the differential of) a character of $\mathbf{A}_P(\mathbb{C})$.

Convention II. — In the following we fix the extension of $I_{\mu,\lambda}$ to a $(\mathfrak{g}, \widetilde{K})$-module to be the one associated to the interwining operator $A_G = \operatorname{ind}_{\mathbf{P}(\mathbb{C})}^{\mathbf{G}(\mathbb{C})}(A_M \otimes 1)$, where A_M is chosen according to Convention I.

Let $K_M = K \cap {}^0\mathbf{M}(\mathbb{C})$. Since σ stabilizes ${}^0\mathbf{M}(\mathbb{C})$, μ, etc. it follows from Frobenius reciprocity and (3.6.2) that we have:

$$\operatorname{Lef}'(\sigma, F, I_{\mu,\lambda}) = \operatorname{trace}\Big(\sigma \mid \Big[\det{}'[1 - (\mathfrak{g}/\mathfrak{k})^*] \otimes F \otimes \pi_{\mu,0}^{{}^0\mathbf{M}(\mathbb{C})}\Big]^{K_M}\Big). \tag{6.5.1}$$

Write

$$\mathfrak{g}/\mathfrak{k} = {}^0\mathfrak{m}/\mathfrak{k}_M \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

It follows from (3.6.3) that — as a $\widetilde{K}_M$-module — we have:

$$\begin{aligned} \det{}'[1 - (\mathfrak{g}/\mathfrak{k})^*] = \det[1 - ({}^0\mathfrak{m}/\mathfrak{k}_M)^*] \otimes \det{}'[1 - \mathfrak{a}^* \oplus \mathfrak{n}^*] \\ \oplus \det{}'[1 - ({}^0\mathfrak{m}/\mathfrak{k}_M)^*] \otimes \det[1 - \mathfrak{a}^* \oplus \mathfrak{n}^*], \end{aligned}$$

with

$$\det[1 - \mathfrak{a}^* \oplus \mathfrak{n}^*] = \det[1 - \mathfrak{a}^*] \otimes \det[1 - \mathfrak{n}^*] \tag{6.5.2}$$

[5] Beware not to confuse the half-sum with the finite dimensional representation...

and

$$\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]=\det[1-\mathfrak{n}^*]\otimes\det'[1-\mathfrak{a}^*]\oplus\det'[1-\mathfrak{n}^*]\otimes\det[1-\mathfrak{a}^*].$$

Now two simple lemmas:

6.6. Lemma

(1) *We have* $\det[1-\mathfrak{a}^*]=0$ *as a virtual* $\widetilde{K}_M$*-module unless* $\mathfrak{a}^\sigma=0$.
(2) *We have* $\det'[1-\mathfrak{a}^*]=0$ *as a virtual* $\widetilde{K}_M$*-module unless* $\dim\mathfrak{a}^\sigma\leqslant 1$.

Proof. — For any $\delta\in\widetilde{K}_M$,

$$\operatorname{trace}(\delta|\det[1-\mathfrak{a}^*])=\det(1-\delta|\mathfrak{a}^*),\operatorname{trace}(\delta|\det'[1-\mathfrak{a}^*])=\frac{d}{dt}\Big|_{t=1}\det(t\cdot 1-\delta|\mathfrak{a}^*),$$

cf. Section 3.1. Write $\delta=\sigma k$, with $k\in K_M$.

For any $X\in\mathfrak{a}^*$,

$$\delta X=\sigma kX=\sigma X$$

since K_M centralizes $\mathfrak{a}^*$. Thus,

$$\dim\{+1\text{-eigenspace of }\delta\}\geqslant\dim(\mathfrak{a}^*)^\sigma.$$

In particular,

(1) if $\dim\mathfrak{a}^\sigma>0$, then
$$\det(1-\delta|\mathfrak{a}^*)=0.$$
(2) if $\dim\mathfrak{a}^\sigma>1$, then $\det(t\cdot 1-\delta|\mathfrak{a}^*)$ vanishes to order at least 2 at $t=1$, whence
$$\frac{d}{dt}\Big|_{t=1}\det(t\cdot 1-\delta|\mathfrak{a}^*)=0.\qquad\square$$

6.7. Lemma. — *Let V be a finite dimensional $\widetilde{K}_M$-module and τ any admissible $\widetilde{K}_M$-module, i.e., a $\widetilde{K}_M$-module all of whose K_M-isotypic subspaces are finite dimensional. Suppose V is virtually trivial. Then $[V\otimes\tau]^{K_M}$ is finite dimensional and*

$$\operatorname{trace}(\sigma|[V\otimes\tau]^{K_M})=0.$$

Proof. — Finite dimensionality is immediate since τ is admissible. Let ζ be a finite dimensional subrepresentation of τ such that $[V\otimes\tau]^{K_M}=[V\otimes\zeta]^{K_M}$.

Since K_M is compact, taking K_M-invariants is an exact functor from the category of finite dimensional K_M-modules to the category of finite dimensional σ-modules. Virtually trivial $\widetilde{K}_M$-modules therefore map to virtually trivial σ-modules. Thus, $[V\otimes\zeta]^{K_M}$ is virtually trivial. In particular,

$$\operatorname{trace}(\sigma|[V\otimes\tau]^{K_M})=\operatorname{trace}(\sigma|[V\otimes\zeta]^{K_M})=0.\qquad\square$$

In particular, we conclude that (6.5.1) is zero unless $\dim\mathfrak{a}\leqslant 1$. In the following we compute (6.5.1) in the two remaining cases.

6.8. Computation of (6.5.1) when $\dim\mathfrak{a}=1$. — Assume that $\dim\mathfrak{a}=1$. It then follows from Lemmas 6.6 and 6.7 that

$$\operatorname{trace}(\sigma \mid \det'[1-(\mathfrak{g}/\mathfrak{k})^*]) = \operatorname{trace}(\sigma \mid \det[1-({}^0\mathfrak{m}/\mathfrak{k}_M)^*]\otimes\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]).$$

We can therefore compute

$$\begin{aligned}
&\operatorname{Lef}'(\sigma,F,I_{\mu,\lambda})\\
&\quad= \operatorname{trace}\Big(\sigma \mid \Big[\det[1-({}^0\mathfrak{m}/\mathfrak{k}_M)^*]\otimes\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]\otimes F\otimes\pi_{\mu,0}^{{}^0\mathbf{M}(\mathbb{C})}\Big]^{K_M}\Big) \qquad (6.8.1)\\
&\quad= \operatorname{Lef}(\sigma,\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]\otimes F, I_\mu^{{}^0\mathbf{M}(\mathbb{C})}).
\end{aligned}$$

For each $w\in W$ we let ν_w be the restriction of $w(\rho+\nu)$ to $\mathfrak{t}$. Let $[W_{K_M}\backslash W]$ be the set of $w\in W_U$ such that ν_w is dominant as a weight on $\mathfrak{t}$ (with respect to the roots of $\mathfrak{t}$ on $\mathfrak{k}_M$), i.e.,

$$[W_{K_M}\backslash W] = \{w\in W \,:\, \langle\nu_w,\beta\rangle\geqslant 0 \text{ for all } \beta\in\Delta^+(\mathfrak{t},\mathfrak{k}_M)\}.$$

This is therefore a set of coset representatives for W_{K_M} in W.

6.9. Proposition. — *Assume* $\dim\mathfrak{a}=1$. *Then we have:*

$$\operatorname{Lef}'(\sigma,F,I_{\mu,\lambda}) = \begin{cases}\operatorname{sgn}(w)2^{\dim\mathfrak{t}} & \textit{if } \mu=2\nu_w \textit{ for some } w\in[W_{K_M}\backslash W],\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. — We shall apply Proposition 6.4 to the twisted space associated to ${}^0\mathbf{M}(\mathbb{C})$ to compute (6.8.1). To do so directly, we would need to decompose the virtual representation

$$\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]\otimes F \qquad (6.9.1)$$

into irreducibles. This can be done by hand by reducing to the cases where $\mathbf{G}$ is simple of type $\mathrm{SO}(p,p)$ with p odd, or of type $\mathrm{SL}(3)$. Instead we note that Proposition 6.4 implies that only the essential σ-stable subrepresentations of (6.9.1) contribute to the final expression in (6.8.1). We may therefore reduce to considering the virtual representation

$$\begin{aligned}
&(\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes(\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*])^\sigma)\otimes(F_0\otimes F_0^\sigma)\\
&\qquad\qquad= (\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0)\otimes(\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0)^\sigma.
\end{aligned}$$

Here, we have realized the $\widetilde{K}_M$-representations $\mathfrak{a}$ and $\mathfrak{n}$ as representations in $\mathfrak{a}_0\otimes\mathfrak{a}_0^\sigma$ and $\mathfrak{n}_0\otimes\mathfrak{n}_0^\sigma$, resp.

Now, it follows from Proposition 6.4 that $\operatorname{Lef}(\sigma,F,I_{\mu,\lambda})=0$ unless $\mu=2\mu_0$, where $\mu_0-\rho$ is the highest weight of a finite dimensional representation of ${}^0\mathbf{M}(\mathbb{R})$. Equation (6.8.1) therefore implies that $\operatorname{Lef}'(\sigma,F,I_{\mu,\lambda})$ is equal to $2^{\dim\mathfrak{t}}$ times the number N of irreducible σ-stable ${}^0\mathbf{M}(\mathbb{C})$-subrepresentations of $\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]\otimes F$ with the same infinitesimal character as $I_{2\mu_0,0}$. Since $\det'[1-\mathfrak{a}^*\oplus\mathfrak{n}^*]\otimes F$ decomposes as

$$(\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0)\otimes(\det'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0)^\sigma,$$

the number N is the number of irreducible ${}^0\mathbf{M}(\mathbb{R})$-subrepresentations of

$$\det{}'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0$$

with infinitesimal character equal to that of θ_0.

Next, we use that, if θ_0 denotes the discrete series of ${}^0\mathbf{M}(\mathbb{R})$ with infinitesimal character μ_0 and H_0 a finite dimensional representation of ${}^0\mathbf{M}(\mathbb{R})$ of highest weight ν, then the (untwisted) Euler-Poincaré characteristic

$$\chi(H_0,\theta_0)=\dim\bigl[\theta_0\otimes\det[1-({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})^*]\otimes H_0\bigr]^{K_M^\sigma}$$

of the $({}^0\mathfrak{m},K_M^\sigma)$-complex of $\theta_0\otimes H_0$ is given by the following formula (see e.g. [28, Prop. 3.1 & 3.2]):

$$\chi(H_0,\theta_0)=\begin{cases}(-1)^{\frac12\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})} & \text{if } w\mu_0=\nu+\rho\quad (w\in W_M),\\ 0 & \text{otherwise.}\end{cases}$$

We can extend $\chi(\cdot,\theta_0)$ to any virtual representation. Applying this to

$$H_0=\det{}'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0,$$

we conclude that the number N is equal to

$$(-1)^{\frac12\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})}\chi(\det{}'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0,\theta_0).$$

Now by definition of the Euler-Poincaré characteristic the latter is equal to

$$(-1)^{\frac12\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})}\dim\bigl[\theta_0\otimes\det[1-({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})]\otimes\det{}'[1-\mathfrak{a}_0^*\oplus\mathfrak{n}_0^*]\otimes F_0\bigr]^{K_M^\sigma},$$

which, as in (6.8.1) (but in an untwisted setting), is equal to

$$(-1)^{\frac12\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})+1}\dim\bigl[I_{\mu_0}\otimes\det{}'[1-(\mathfrak{g}(\mathbb{R})/\mathfrak{k}_{\mathbb{R}})^*]\otimes F_0\bigr]^{K^\sigma}.$$

The computation of the latter is made in [3, §5.6]:

$$\begin{aligned}&\dim[I_{\mu_0}\otimes\det{}'[1-(\mathfrak{g}(\mathbb{R})/\mathfrak{k}_{\mathbb{R}})^*]\otimes F_0]^{K^\sigma}\\ &\qquad=\begin{cases}(-1)^{\frac12\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})+1}\operatorname{sgn}(w) & \text{if } \mu=2\nu_w \text{ for some } w\in[W_{K_M}\backslash W],\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We conclude that

$$\mathrm{Lef}'(\sigma,F,I_{\mu,\lambda})=2^{\dim\mathfrak{t}}N=\begin{cases}\operatorname{sgn}(w)2^{\dim\mathfrak{t}} & \text{if } \mu=2\nu_w \text{ for some } w\in[W_{K_M}\backslash W],\\ 0 & \text{otherwise.}\end{cases}$$

□

It is natural to introduce the untwisted analogue of $\mathrm{Lef}'(\sigma,F,I_{\mu,\lambda})$. With the notation of Proposition 6.9 we shall therefore denote by $\det{}'(F_0,I_{\mu_0})$ the dimension

$$\dim[I_{\mu_0}\otimes\det{}'[1-(\mathfrak{g}(\mathbb{R})/\mathfrak{k}_{\mathbb{R}})^*]\otimes F_0]^{K^\sigma}.$$

Remarks

(1) The proof above and the transfer of infinitesimal characters under base change shows that

$$\text{(twisted heat kernel for } F)(2t) \xrightarrow{\text{transfer}} 2^{\dim \mathfrak{t}}(-1)^{\frac{1}{2}\dim({}^0\mathfrak{m}(\mathbb{R})/\mathfrak{k}_{M,\mathbb{R}})} \times \text{(heat kernel for } F_0)(t)$$

for base change $\mathbb{C}/\mathbb{R}$.

(2) Note that this base change calculation includes, as a special case, that of a product $\mathbf{G} = \mathbf{G}' \times \mathbf{G}'$. But we've worked out separately that

$$\mathrm{Lef}'(\sigma, F_0 \otimes F_0, \pi \otimes \pi) = 2\det{}'(F_0, \pi).$$

There is no contradiction here: if either side is non-zero then $\dim \mathfrak{a} = 1$, but in that special case $\dim \mathfrak{a} = \dim \mathfrak{t}$ since the real group is in fact a complex group.

6.10. Computation of (6.5.1) when $\dim \mathfrak{a} = 0$. — We now assume that $\dim \mathfrak{a} = 0$ and follow an observation made by Müller and Pfaff [27]. In that case $\mathbf{M} = \mathbf{G}$, $K_M = K$ and $\pi_{\mu,\lambda} = \pi_{\mu,0}$ is σ-discrete. Now $\dim \mathfrak{g}(\mathbb{C})/\mathfrak{k}$ equals $2d$, where d is the dimension of the symmetric space associated to $\mathbf{G}(\mathbb{R})$. Note that — as $\widetilde{K}$-modules — we have

$$\wedge^i(\mathfrak{g}/\mathfrak{k})^* \cong \wedge^{2d-i}(\mathfrak{g}/\mathfrak{k})^*, \quad i = 0, \ldots, 2d.$$

It follows that as $\widetilde{K}$-representations we have:

$$\det{}'[1 - (\mathfrak{g}/\mathfrak{k})^*] = d\det[1 - (\mathfrak{g}/\mathfrak{k})^*].$$

This implies that

$$\mathrm{Lef}'(\sigma, F, I_{\mu,0}) = d\ \mathrm{Lef}(\sigma, F, I_{\mu,0}).$$

Proposition 6.4 therefore implies:

6.11. Proposition. — *Let π_μ be a σ-discrete representation of G. Then we have:*

$$\mathrm{Lef}'(\sigma, F, I_\mu) = \begin{cases} 2^{\dim \mathfrak{t}} \dim(\mathfrak{g}^0/\mathfrak{k}^0) & \text{if } w\mu = 2(\nu+\rho)_{|\mathfrak{t}} \quad (w \in W), \\ 0 & \text{otherwise.} \end{cases}$$

6.12. Proof of Lemma 4.10. — If ϕ is any smooth compactly supported function on G, Bouaziz [7] shows that

$$\int_{G^\sigma \backslash G} \phi(x^{-1}\sigma x)\, d\dot{x} = \phi^G(e), \tag{6.12.1}$$

where $\phi^G \in C_c^\infty(\mathbf{G}(\mathbb{R}))$ is the transfer of ϕ. Now we can use the Plancherel theorem of Herb and Wolf [16] (as in [30, Prop. 4.2.14]) and get

$$\phi^G(e) = \sum_{\pi \text{ discrete}} d(\pi)\,\mathrm{trace}\,\pi(\phi^G) + \int_{\text{tempered}} \mathrm{trace}\,\pi(\phi^G) d\pi.$$

We can group the terms into stable terms since all terms in a L-packet have the same Plancherel measure [15]. We write π_φ for the sum of the representations in an L-packet φ and $d\pi_\varphi = d\pi$ for the corresponding measure. We then obtain

$$\phi^G(e) = \sum_{\substack{\text{elliptic}\\ L-\text{packets } \varphi}} d(\varphi)\,\mathrm{trace}\,\pi_\varphi(\phi^G) + \int_{\substack{\text{non elliptic}\\ \text{bounded } L-\text{packets } \varphi}} \mathrm{trace}\,\pi_\varphi(\phi^G)d\pi_\varphi.$$

Now we use transfer again. Indeed, Clozel [13] shows that if φ is a tempered L-packet and $\widetilde{\pi}_\varphi$ the sum of the twisted representations of $\widetilde{G}$ associated to φ by base-change, we have:

$$\mathrm{trace}\,\pi_\varphi(\phi^G) = \mathrm{trace}\,\widetilde{\pi}_\varphi(\phi).$$

We conclude:

$$\begin{aligned}(6.12.2)\quad &\int_{G^\sigma\backslash G} \phi(x^{-1}\sigma x)\,d\dot{x}\\ &\qquad = \sum_{\substack{\text{elliptic}\\ L-\text{packets } \varphi}} d(\varphi)\,\mathrm{trace}\,\widetilde{\pi}_\varphi(\phi) + \int_{\substack{\text{non elliptic}\\ \text{bounded } L-\text{packets } \varphi}} \mathrm{trace}\,\widetilde{\pi}_\varphi(\phi)d\pi_\varphi.\end{aligned}$$

We want to apply this to the function $\phi = k_t^{\rho,\sigma}$. To do so we will use

6.13. Lemma. — *Equation* (6.12.2) *holds for functions ϕ in the Harish-Chandra Schwartz space.*

Proof. — We first note we have already checked (in Section 4.8) that the distribution

$$\phi \longmapsto \int_{G^\sigma\backslash G} \phi(x^{-1}\sigma x)\,d\dot{x}$$

extends continuously to the Harish-Chandra Schwartz space, i.e., it defines a *tempered* distribution. Now, for ϕ compactly supported, Bouaziz [7, Th. 4.3] proves that we have (recall that we suppose that $H^1(\sigma, G) = \{1\}$):

$$(6.13.1)\quad \int_{G^\sigma\backslash G} \phi(x^{-1}\sigma x)\,d\dot{x} = \int_{(\mathfrak{g}_a^*/G)^\sigma} \Bigg(\frac{1}{2}\sum_{\tau\in X(f)} Q_\sigma(f,\tau)\,\mathrm{trace}\,\Pi_{f,\tau}(\phi)\Bigg) dm(G\cdot f).$$

We refer to [7] for all undefined notations and simply note that

– the $\Pi_{f,\tau}$ are tempered (twisted) representation, and
– if ϕ belongs to the Harish-Chandra Schwartz space, the (twisted) characters

$$\Theta_{f,\tau}(\phi) = \mathrm{trace}\,\Pi_{f,\tau}(\phi)$$

define rapidly decreasing functions of f.

Bouaziz does not explicitly compute the function $Q_\sigma(f,\tau)$ but proves however that it grows at most polynomially in f. It therefore follows that the distribution defined by the right hand side of (6.13.1) also extends continuously to the Harish-Chandra

Schwartz space.[6] We conclude that (6.13.1) still holds when ϕ belongs to the Harish-Chandra Schwartz space.

We may now group the characters $\Theta_{f,\tau}$ into finite packets to get (all) stable tempered characters, as in [7, §7.3] and denoted $\widetilde{\Theta}_\lambda$ there. Then the right hand side of (6.13.1) becomes

$$\sum_{\mathfrak{a}\in\mathrm{Car}(\mathfrak{g}^0)/G^0} 2^{-\frac{1}{2}(\dim G^\sigma+\mathrm{rank}\, G^\sigma)}|W(G,\mathfrak{a})|^{-1}\int_{\mathfrak{a}^*} p_{\sigma,\sigma}(\lambda)\Pi^\sigma_{\mathfrak{g}^0}(\lambda)\widetilde{\Theta}_\lambda(\phi)d\eta_{\mathfrak{a}}(\lambda).$$

Here again, we refer to [7] for notations. Then [7, Eq. (3), (4), (5) and (6) p. 287] imply that the right hand side of (6.13.1) is equal to

$$\sum_{\mathfrak{a}\in\mathrm{Car}(\mathfrak{g}^0)/G^0}|W(G,\mathfrak{a})|^{-1}\int_{\mathfrak{a}^*} p_{1,1}(\lambda)\Pi_{\mathfrak{g}^0/\mathfrak{a}}(\lambda)\widetilde{\Theta}_{2\lambda}(\phi)d\eta_{\mathfrak{a}}(\lambda).$$

Here, the stable character $\widetilde{\Theta}_{2\lambda}$ is the transfer of the character of a tempered packet, see [7, 7.3(a)]) that is parametrized by λ there. Finally in this parametrization the measure against which we integrate $\widetilde{\Theta}_{2\lambda}(\phi)$ can be identified with the Plancherel measure (see the very beginning of the proof of [7, Th. 7.4]) and we conclude that (6.12.2) extends to the Harish-Chandra Schwartz space. □

We can now conclude the proof of Lemma 4.10. It follows from Lemma 6.13 that Equation (6.12.2) holds with $\phi = k_t^{\rho,\sigma}$. Using Lemma 3.3 and Propositions 6.9 and 6.11 we can express each twisted trace trace $\widetilde{\pi}_\varphi(\phi)$ as non-twisted trace. Moreover: Proposition 6.11 implies that only representations with the same infinitesimal character as F_0 can contribute to the (finite) first sum. Since $(\widetilde{\rho}, F)$ is supposed to be strongly acyclic, it follows that the elliptic L-packets do not contribute. We therefore conclude that

$$\int_{G^\sigma\backslash G} k_t^{\rho,\sigma}(x^{-1}\sigma x)\, d\dot{x} = 2^{\dim \mathfrak{t}}\int_{\mathrm{tempered}} e^{-t(\Lambda_\rho-\Lambda_\pi)}\det{}'(F_0,\pi)d\pi. \tag{6.13.2}$$

Lemma 4.10 therefore follows from the untwisted case for which we refer to [3].

We furthermore deduce from (6.13.2) (and the computation in the untwisted case) the following theorem.

6.14. Theorem. — *We have:*

$$t_X^{(2)\sigma}(\widetilde{\rho}) = 2^{\dim \mathfrak{t}} t^{(2)}_{X^\sigma}(\rho).$$

Similarly:

6.15. Theorem. — *We have:*

$$\mathrm{Lef}_X^{(2)\sigma}(\widetilde{\rho}) = 2^{\dim \mathfrak{t}}\cdot \chi^{(2)}_{X^\sigma}(\rho).$$

[6] In fact, we need that (twisted) tempered characters are rapidly decreasing in the parameters "Schwartz-uniformly" in ϕ. But this holds because of known properties of discrete series characters combined with the fact that the constant term operator $\phi\mapsto\phi^{(P)}$ are all Schwartz-continuous.

Remark. — It follows from Theorems 5.2 and 6.15 that if $\delta(G^\sigma)=0$ then $\mathrm{Lef}^{(2)\sigma}_X(\widetilde{\rho})$ is non-zero. Proposition 1.2 of the Introduction therefore follows from the limit formula proved in Corollary 4.8.

7. General base change

7.1. Twisted torsion of product automorphisms. — Let $\mathbf{G}/\mathbb{R}$ be semisimple, σ be an automorphism of $\mathbf{G}$, and ρ a σ-equivariant representation of $\mathbf{G}$. Let $\mathbf{G}'/\mathbb{R}$ be semisimple, σ' be an automorphism of $\mathbf{G}'$, and ρ' a σ'-equivariant representation of $\mathbf{G}'$.

7.2. Lemma. — *There is an equality*

$$t^{(2)\sigma\times\sigma'}_{X_{G\times G'}}(\rho\boxtimes\rho')=t^{(2)\sigma}_{X_G}(\rho)\cdot\mathrm{Lef}^{(2)\sigma'}_{X_{G'}}(\rho')+\mathrm{Lef}^{(2)\sigma}_{X_G}(\rho)\cdot t^{(2)\sigma'}_{X_{G'}}(\rho').$$

Proof. — Let M,M' be compact Riemannian manifolds together with equivariant metrized local systems $L\to M$ and $L'\to M'$. Lück [25, Prop. 1.32] proves that

$$t^{\sigma\times\sigma'}(M\times M';L\boxtimes L')=t^\sigma(M,L)\cdot\mathrm{Lef}(\sigma',M',L')+\mathrm{Lef}(\sigma,M,L)\cdot t^{\sigma'}(M',L').$$

Furthermore, Theorem 4.11 shows that

$$t^{(2)\sigma\times\sigma}_{X_{G\times G'}}(\rho\boxtimes\rho')=\lim_{n\to\infty}\frac{\log\tau^{\sigma\times\sigma'}(\Upsilon_n\backslash X_{G\times G'})}{\mathrm{vol}(\Upsilon_n\backslash G\times G')}$$

for any sequence of subgroups Υ_n satisfying the hypotheses therein. The sequence $\Upsilon_n=\Gamma_n\times\Gamma'_n$, where Γ_n (resp. Γ'_n) is a chain of σ-stable (resp. σ'-stable) normal subgroups of G (resp. G') with trivial intersection, satisfies the hypotheses of Proposition 4.6 and Theorem 4.11. Therefore,

$$\begin{aligned}t^{(2)\sigma\times\sigma'}_{X_{G\times G'}}(\rho\boxtimes\rho')&=\lim_{n\to\infty}\frac{t^\sigma(\Gamma_n\backslash X_G,\rho)}{\mathrm{vol}(\Gamma_n^\sigma\backslash G^\sigma)}\cdot\frac{\mathrm{Lef}(\sigma',\Gamma'_n\backslash X_{G'},\rho')}{\mathrm{vol}(\Gamma'^{\sigma'}_n\backslash G'^{\sigma'})}\\&\qquad\qquad+\frac{\mathrm{Lef}(\sigma,\Gamma_n\backslash X_G,\rho)}{\mathrm{vol}(\Gamma_n^\sigma\backslash G^\sigma)}\cdot\frac{t^{\sigma'}(\Gamma_n\backslash X_{G'},\rho')}{\mathrm{vol}(\Gamma'^{\sigma'}_n\backslash G'^{\sigma'})}\\&=t^{(2)\sigma}_{X_G}(\rho)\cdot\mathrm{Lef}^{(2)\sigma'}_{X_{G'}}(\rho')+\mathrm{Lef}^{(2)\sigma}_{X_G}(\rho)\cdot t^{(2)\sigma'}_{X_{G'}}(\rho').\end{aligned}$$
□

7.3. — Now let $\mathbb{E}$ be an étale $\mathbb{R}$-algebra; concretely, $\mathbb{E}=\mathbb{R}^r\times\mathbb{C}^s$. Fix $\sigma\in\mathrm{Aut}(\mathbb{E}/\mathbb{R})$. The automorphism σ permutes the factors of $\mathbb{E}$ and so induces a decomposition of the factors of $\mathbb{E}$ into its set of orbits $\mathscr{O}:\mathbb{E}=\prod_{o\in\mathscr{O}}\mathbb{E}_o$. Each orbit is either

(a) a product of real places acted on by cyclic permutation,
(b) a product of complex places acted on by cyclic permutation, or
(c) a single complex place acted on by complex conjugation.

Let $\mathbf{G}$ be a semisimple group over $\mathbb{R}$. Let ρ be a representation of $\mathbf{G}/\mathbb{R}$ and $\widetilde{\rho}_o$ the corresponding representation of $\mathrm{Res}_{\mathbb{E}_o/\mathbb{R}}$. In particular, $\widetilde{\rho}=\rho\otimes\overline{\rho}$ is the corresponding representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}$. The automorphism σ induces a corresponding automorphism of the group $\mathrm{Res}_{\mathbb{E}/\mathbb{R}}\mathbf{G}$. There is a decomposition

$$\mathrm{Res}_{\mathbb{E}/\mathbb{R}}\mathbf{G}=\prod_{o\in\mathscr{O}}\mathrm{Res}_{\mathbb{E}_o/\mathbb{R}}\mathbf{G}$$

with respect to which σ acts as a product automorphism.

– Theorem 5.3 shows that

$$t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = |o| \cdot t^{(2)}_{X_{\mathbf{G}(\mathbb{R})}}(\rho) \quad \text{and} \quad \mathrm{Lef}^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = \chi^{(2)}_{X_{\mathbf{G}(\mathbb{R})}}(\rho)$$

for the orbits of type (a).

– Theorem 5.3 shows that

$$t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = |o| \cdot t^{(2)}_{X_{\mathbf{G}(\mathbb{C})}}(\widetilde{\rho}) \quad \text{and} \quad \mathrm{Lef}^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = \chi^{(2)}_{X_{\mathbf{G}(\mathbb{C})}}(\widetilde{\rho})$$

for the orbits of type (b).

– Theorem 6.14 proves that

$$t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = 2^{\dim \mathfrak{t}} t^{(2)}_{X_{\mathbf{G}(\mathbb{R})}}(\rho) \quad \text{and} \quad \mathrm{Lef}^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o) = \chi^{(2)}_{X_{\mathbf{G}(\mathbb{R})}}(\rho)$$

for the orbits of type (c).

The aggregate of these three examples, together with Theorem 7.2, enables us to compute the twisted L^2-torsion for arbitrary base change.

7.4. Theorem. — *We have*

$$t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E})}}(\widetilde{\rho}_{\mathbb{E}}) \neq 0$$

if and only if $\delta(\mathbf{G}(\mathbb{E})^\sigma) = 1$.

Proof. — Suppose that there are n orbits. Using Lemma 7.2, we expand $t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E})}}(\widetilde{\rho}_{\mathbb{E}})$ as a sum of n terms. Each summand is a product of $\mathrm{Lef}^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o)$ for $n-1$ of the orbits o and of $t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_o)}}(\widetilde{\rho}_o)$ for the remaining orbit o. Thus, exactly $n-1$ of the $\mathrm{Lef}^{(2)\sigma}$'s must be non-zero and the remaining $t^{(2)\sigma}$ must be non-zero; say $t^{(2)\sigma}_{X_{\mathbf{G}(\mathbb{E}_{o^*})}}(\rho_{o^*}) \neq 0$. But by the preceding computations relating $t^{(2)\sigma}$ and $\mathrm{Lef}^{(2)\sigma}$ to their untwisted analogues, this is possible if and only if $\delta(\mathbf{G}(\mathbb{E}_{o^*})^\sigma) = 1$ and $\delta(\mathbf{G}(\mathbb{E}_o)^\sigma) = 0$ for all $o \neq o^*$. This is equivalent to $\delta(\mathbf{G}(\mathbb{E})^\sigma) = 1$. □

8. Application to torsion in cohomology

8.1. Generalities on Reidemeister torsion and the Cheeger-Müller theorem

Let $A^\bullet$ be a finite chain complex situated in degree $\geqslant 0$ of K-vector spaces for a field K. Suppose the chain groups A^i and the cohomology groups $H^i(A^\bullet)$ are equipped with volume forms, i.e., with non-zero elements $\omega_i \in \det(A^i)^*$ and $\mu_i \in (\det(H^i(A^\bullet))^*$. These define elements ω in $\det(A^\bullet)^*$ and μ in $\det(H(A^\bullet))^*$, where

$$\det(A^\bullet) := \det(A_0) \otimes \det(A_1)^{-1} \otimes \det(A_2) \otimes \cdots$$

and similarly for $\det(H(A^\bullet))$.

There is a natural isomorphism [18, §1]

$$\det(A^\bullet) \otimes \det(H(A^\bullet)) \cong K. \tag{8.1.1}$$

We let $s_{A^\bullet}$ denote the preimage of 1 under the above isomorphism.

8.2. Definition (Reidemeister torsion of a complex with volume forms)

The Reidemeister torsion $RT(A^\bullet, \omega_\bullet, \mu_\bullet)$ is defined to be

$$\omega \otimes \mu^{-1}(s_{A^\bullet}).$$

It is readily checked that if $\mu'_i = c_i\mu_i$ for some non-zero constants c_i, then

$$RT(A^\bullet, \omega, \mu') = \frac{c_0 c_2 \cdots}{c_1 c_3 \cdots} \cdot RT(A^\bullet, \omega, \mu). \tag{8.2.1}$$

8.3. Example. — Suppose $K = \mathbb{C}$ and $A^\bullet = C^\bullet \otimes \mathbb{C}$ for some finite complex of free abelian groups $C^\bullet$. A $\mathbb{Z}$-basis $(e_1, \ldots, e_n)$ of C^i determines a volume form ω_i on $A^i = C^i \otimes_{\mathbb{Z}} \mathbb{C}$ by the formula

$$\omega_i(e_1 \wedge \cdots \wedge e_n) = 1.$$

The volume form ω_i is well-defined up to sign. We endow $A^\bullet$ with such a volume form $\omega_{\mathbb{Z}} \in \det(A^\bullet)^*$. Using that $H^i(A^\bullet) = H^i(C^\bullet)_{\mathbb{C}}$ we can similarly endow $H(A^\bullet)$ with a volume form $\mu_{\mathbb{Z}} \in \det(H(A^\bullet))^*$. Then

$$|RT(A^\bullet, \omega_{\mathbb{Z}}, \mu_{\mathbb{Z}})| = \frac{|H^1(C^\bullet)_{\mathrm{tors}}| \cdot |H^3(C^\bullet)_{\mathrm{tors}}| \cdots}{|H^0(C^\bullet)_{\mathrm{tors}}| \cdot |H^2(C^\bullet)_{\mathrm{tors}}| \cdots}.$$

8.4. Definition. — Fix a triangulation T of a Riemannian manifold (M, g) together with a metrized local system of free abelian groups $L \to M$. Let $C^i(M, L; T)$ be the corresponding cochain (free abelian) groups. We can endow $A^\bullet := C^\bullet(M, L; T) \otimes \mathbb{C}$ with a combinatorial volume form $\omega_{\mathbb{Z}}$ as defined in Example 8.3. Identifying each $H^i(M, L_{\mathbb{C}})$ with the vector space of harmonic $L_{\mathbb{C}}$-valued i-forms on M we define a volume form μ_g on $H^\bullet(M, L_{\mathbb{C}}) = H(A^\bullet)$. We finally define

$$RT(M, L) := RT(A^\bullet, \omega_{\mathbb{Z}}, \mu_g).$$

It follows from Example 8.3 and (8.2.1) that we have:

$$|RT(M, L)| = \frac{|H^1(M, L)_{\mathrm{tors}}| \cdot |H^3(M, L)_{\mathrm{tors}}| \cdots}{|H^0(M, L)_{\mathrm{tors}}| \cdot |H^2(M, L)_{\mathrm{tors}}| \cdots} \times \frac{R^0(M, L) R^2(M, L) \cdots}{R^1(M, L) R^3(M, L) \cdots}, \tag{8.4.1}$$

where $R^i(M, L)$ is the volume $\mathrm{vol}(H^i(M, L)_{\mathrm{free}})$, with respect to the volume forms obtained by identifying $H^i(M, L_{\mathbb{C}})$ with the space of harmonic L-valued i-forms on M.

Note that it follows in particular from (8.4.1) that $|RT(M, L)|$ does not depend on the triangulation T. The following beautiful theorem relates $|RT(M, L)|$ to the analytic torsion $T_M(L) = \exp(t_M(L))$ that we have already considered in the particular case of locally symmetric spaces[7], see [26] for the general definition.

8.5. Theorem (Cheeger-Müller theorem for unimodular local systems [26])

Let $L \to M$ be a unimodular local system over a compact Riemannian manifold. There is an equality

$$T_M(L) = |RT(M, L)|.$$

[7] In fact we have more generally considered the *twisted* analytic torsion that we deal with below.

8.6. Twisted local system over locally symmetric spaces. — Let $\mathbf{G}$ be a connected semisimple quasi-split algebraic group defined over $\mathbb{R}$. Let $\mathbb{E}$ be an étale $\mathbb{R}$-algebra such that $\mathbb{E}/\mathbb{R}$ is a cyclic Galois extension with Galois group generated by $\sigma \in \mathrm{Aut}(\mathbb{E}/\mathbb{R})$. The automorphism σ induces a corresponding automorphism of the group G of real points of $\mathrm{Res}_{\mathbb{E}/\mathbb{R}}\,\mathbf{G}$. We assume that Condition (2.4.1) holds, namely: $H^1(\sigma, G) = \{1\}$.

Choose a Cartan involution θ of G that commutes with σ and denote by K the group of fixed points of θ in G and let $X = G/K$ be the associated symmetric space and. The involution σ acts on G and X; we denote by G^σ and X^σ the corresponding sets of fixed points.

As above we denote by $\widetilde{G}$ the twisted space $G \rtimes \sigma$. Now, let Γ be a torsion free cocompact lattice of G that is σ-stable and let $(\widetilde{\rho}, F)$ be a complex finite dimensional σ-stable irreducible representation of $\widetilde{G}$ defined over $\mathbb{R}$ such that

(1) $\rho(\Gamma)$ stabilizes some fixed lattice $\mathscr{O}$ in the real points of F ;

(2) the representation $\widetilde{\rho}$ is strongly twisted acyclic (see Section 2.6).

The group Γ acts on X and diagonally on $X \times \mathscr{O}$ (through the representation ρ on the second factor). We let

$$\mathscr{M} = \Gamma\backslash X \quad \text{and} \quad \mathscr{L} = \Gamma\backslash(X \times \mathscr{O})$$

be the corresponding quotients. Projection on the first factor gives a unimodular local system $\mathscr{L} \to \mathscr{M}$ of free abelian groups; it is moreover equivariant with respect to an automorphism of $\mathscr{M}$ of finite order.[8] From now on we shall furthermore assume that the order p of σ is *prime*.

We shall consider a family of covering manifolds $\mathscr{M}_n$ associated to a sequence $\{\Gamma_n\}$ of finite index σ-stable subgroups of Γ that satisfies the hypothesis of Section 4.3.

8.7. Equivariant Reidemeister torsion. — Let $P(x) = x^{p-1} + x^{p-2} + \cdots + 1$. For a polynomial $h \in \mathbb{Z}[x]$ and a $\mathbb{Z}[\sigma]$-module A, we define $A^{h(\sigma)} := \{a \in A : h(\sigma) \cdot a = 0\}$. We shall denote by $A[p^{-1}]$ the localization $S^{-1}A$, where S is the multiplicative subset $\{p^n : n \in \mathbb{Z}\} \subset \mathbb{Z} \subset \mathbb{Z}[\sigma]$ and let $A[p^\infty] = \{a \in A : p^n a = 0 \text{ for some } n \geqslant 0\}$, the p-power torsion subgroup of A. Finally if K is a field we set $\mathscr{L}_K := \mathscr{L} \otimes_{\mathbb{Z}} K$.

In particular, $\mathscr{L}_{\mathbb{C}}$ defines a flat complex bundle on $\mathscr{M}$ and the action of σ on $\mathscr{M}$ lifts to the flat vector bundle $\mathscr{L}_{\mathbb{C}}$. We shall apply results of Bismut and Zhang to relate equivariant combinatorial and analytical torsions of $\mathscr{L}_{\mathbb{C}} \to \mathscr{M}$.

First recall that it is a general result of Wasserman that there exists a σ-invariant Morse function $f : \mathscr{M} \to \mathbb{R}$. We shall in fact work with particular choices of Morse functions on the $\mathscr{M}_n$'s. By [17] there exists an equivariant CW-triangulation on $\mathscr{M}$. It lifts to an equivariant CW-triangulation on each $\mathscr{M}_n$. By a standard construction there correspond to these triangulations natural Morse functions $f_n : \mathscr{M}_n \to \mathbb{R}$ such that the set of critical points of f_n is exactly the set of barycenters of the simplexes of the CW-triangulation of $\mathscr{M}_n$. One verifies easily that these constructions can be made

[8] We shall use the notation $\mathscr{M}$ for manifolds with an involution and the notation M when there is no involution.

Γ/Γ_n-equivariantly and that the resulting functions f_n can be made σ-invariant. The proof of [5, Th. 1.10] moreover implies that one can construct these f_n's such that:

$$\text{for any critical point } x \in \mathrm{Crit}(f_n) \cap \mathscr{M}_n^\sigma, \\ \text{the Hessian } d^2 f_n(x) \text{ is positive definite on } N_x. \tag{8.7.1}$$

Here, N denotes the normal bundle to $\mathscr{M}_n^\sigma$ in $\mathscr{M}_n$.

It then follows from [5, Th. 1.8] that one can modify the locally symmetric metric g on $\mathscr{M}_n$ to get a metric g' which equals g in a neighborhood of all critical points of f_n in such a way that the corresponding gradient vector field $X_n = \nabla_{g'}(f_n)$ satisfies the Smale transversality condition.

To ease notation we shall now concentrate of $\mathscr{M}$ and f and explain when needed what happens when $(\mathscr{M}, f)$ is replaced by $(\mathscr{M}_n, f_n)$. Let $\mathrm{MS}(\mathscr{M}, \mathscr{L}_\mathbb{C})$ be the Morse-Smale complex [4, §1.6] associated to the σ-invariant Morse function f and the associated invariant transversal gradient vector field X. We endow the chain groups of $\mathrm{MS}(\mathscr{M}, \mathscr{L}_\mathbb{C})^{\sigma-1}$ and $\mathrm{MS}(\mathscr{M}, \mathscr{L}_\mathbb{C})^{P(\sigma)}$ with volume forms induced from the metric on $\mathscr{L}_\mathbb{C}$ and the combinatorial volume forms induced by the unstable cells of the gradient vector field [23, §1.4]. We endow the cohomology groups $H^i(\mathscr{M}, \mathscr{L}_\mathbb{C})^{\sigma-1}$ and $H^i(\mathscr{M}, \mathscr{L}_\mathbb{C})^{P(\sigma)}$ with the metric induced by the L^2-metric on $\mathscr{L}_\mathbb{C}$-valued harmonic i-forms on $\mathscr{M}$ [23, Def. B.4].

8.8. Definition. — The equivariant Reidemeister torsion of the equivariant local system $\mathscr{L}_\mathbb{C} \to \mathscr{M}$ of $\mathbb{C}$-vector spaces is defined by

$$\begin{aligned} \log RT_\sigma(\mathscr{M}, \mathscr{L}_\mathbb{C}; f, X) := \log |RT(\mathrm{MS}(\mathscr{M}, \mathscr{L}_\mathbb{C})^{\sigma-1})| \\ - \frac{1}{p-1} \log |RT(\mathrm{MS}(\mathscr{M}, \mathscr{L}_\mathbb{C})^{P(\sigma)})|. \end{aligned}$$

Remarks

(1) Equivariant Reidemeister torsion a priori depends on the choice of Morse theoretic data; see the remark following Theorem 8.12 for further discussion. However, the discrepancy between equivariant Reidemeister torsion and a purely cohomological quantity can be bounded independently of the Morse theoretic data for local systems endowed with integral structure; see Theorem 8.10 below.

(2) Let G be the finite group generated by σ. We can define a Reidemeister torsion $\log RT_G(\mathscr{M}, \mathscr{L}_\mathbb{C}; f, X)$ with values in the complex representation ring of G. We could then have alternatively defined $\log RT_\sigma(\mathscr{M}, \mathscr{L}_\mathbb{C}; f, X)$ as the trace of σ in $\log RT_G(\mathscr{M}, \mathscr{L}_\mathbb{C}; f, X)$; this viewpoint is closer to that of [5]. Both definitions agree: indeed if C is a finite dimensional $\mathbb{Z}[\sigma]$-module, its complexification $A = C_\mathbb{C}$ decomposes as a direct sum $\oplus_\chi A_\chi$ of isotypical subspaces indexed by characters of $\mathbb{Z}/p\mathbb{Z}$, and we have

$$C^\sigma = A_1 \quad \text{and} \quad C^{P(\sigma)} = \oplus_{\chi \neq 1} A_\chi.$$

We conclude that the trace of σ in C is equal to

$$\dim C^\sigma + (\zeta + \cdots + \zeta^{p-1}) \frac{1}{p-1} \dim C^{P(\sigma)} = \dim C^\sigma - \frac{1}{p-1} \dim C^{P(\sigma)}.$$

(Here, ζ is some primitive pth root of unity.)

8.9. Definition. — Let $R^i(\mathscr{M},\mathscr{L})^{\sigma-1}$ denote the covolume of the lattice $H^i(\mathscr{M},\mathscr{L})^{\sigma-1}$ in the real vector space $H^i(\mathscr{M},\mathscr{L}_{\mathbb{R}})^{\sigma-1}$ with inner product induced by that on harmonic forms. Define $R^i(\mathscr{M},\mathscr{L})^{P(\sigma)}$ similarly. The *equivariant regulator* $R^i_\sigma(\mathscr{M},\mathscr{L})$ is defined to be

$$R^i_\sigma(\mathscr{M},\mathscr{L}) := \frac{R^i(\mathscr{M},\mathscr{L})^{\sigma-1}}{\big(R^i(\mathscr{M},\mathscr{L})^{P(\sigma)}\big)^{1/(p-1)}}.$$

8.10. Theorem (Concrete relationship between twisted RT and cohomology)

Suppose that the fixed point set $\mathscr{M}^\sigma$ has Euler characteristic 0. Then, for an arbitrary choice of invariant Morse function f and invariant transversal gradient vector field X,

$$\begin{aligned}\log & RT_\sigma(\mathscr{M},\mathscr{L}_{\mathbb{C}};f,X)\\ &= -\sum_i(-1)^i\Big(\log\big|H^i(\mathscr{M},\mathscr{L})_{\mathrm{tors}}[p^{-1}]^{\sigma-1}\big| - \frac{1}{p-1}\log\big|H^i(\mathscr{M},\mathscr{L})_{\mathrm{tors}}[p^{-1}]^{P(\sigma)}\big|\Big)\\ &\qquad + \sum_i(-1)^i\log R^i_\sigma(\mathscr{M},\mathscr{L})\\ &\qquad + O\Big(\log|H^*(\mathscr{M},\mathscr{L})_{\mathrm{tors}}[p^\infty]| + \log|H^*(\mathscr{M},\mathscr{L}_{\mathbb{F}_p})| + \log|H^*(\mathscr{M}^\sigma,\mathscr{L}_{\mathbb{F}_p})|\Big).\end{aligned}$$

Proof. — For $\mathscr{L}_{\mathbb{C}}$ acyclic, this is proven in [23, Corollary 3.8]. As described in [23, Prop. B.6], almost exactly the same proof applies even to those $\mathscr{L}$ for which $\mathscr{L}_{\mathbb{C}}$ is not acyclic. $\square$

Remark. — It follows from the proof given in [23] that the implicit constant in $O(\cdot)$ only depends on the dimension of $\mathscr{M}_n$. In particular, it is independent of n.

8.11. The equivariant Cheeger-Müller theorem. — Recall that we have defined in (3.5.1) the twisted analytic torsion of a locally symmetric space equipped with an equivariant, metrized, unimodular local system of complex vector spaces acted on equivariantly and isometrically by σ of finite order. Alternatively this is equal to

$$\log T^\sigma_{(\mathscr{M},g)}(\mathscr{L}_{\mathbb{C}}) := \frac{1}{2}\frac{\partial\theta^{\mathscr{L}_{\mathbb{C}}}}{\partial s}(0)(\sigma),$$

where the function $\theta^{\mathscr{L}_{\mathbb{C}}}(s)(\sigma)$ is defined in [5, Def. 2.2] and g denotes the locally symmetric metric on $\mathscr{M}$. The following theorem will be easily deduced from the deep work of Bismut and Zhang [5].

8.12. Theorem (Equivariant Cheeger-Müller theorem). — *Suppose that X^σ is odd-dimensional. Then we have:*

$$\log T^\sigma_{(\mathscr{M},g)}(\mathscr{L}_{\mathbb{C}}) = \log RT_\sigma(\mathscr{M},\mathscr{L}_{\mathbb{C}};f,X).$$

Proof. — First observe that it follows from [29, Prop. 2.3] that the set of fixed points

$$\mathscr{M}^\sigma = (\Gamma\backslash X)^\sigma$$

is a finite disjoint union of its connected components that are parametrized by $H^1(\sigma,\Gamma)$. Since, under our assumption (2.4.1), the map

$$H^1(\sigma,\Gamma) \longrightarrow H^1(\sigma,G)$$

has obviously trivial image, it moreover follows that each connected component is isometric to $\Gamma^\sigma\backslash X^\sigma$. In particular, it is odd dimensional. Moreover, splitting the complex vector space $F=\oplus_j F^{\alpha_j}$ according to the eigenvalues of σ yields a decomposition of the complex vector bundle $\mathscr{L}_{\mathbb{C}}$ over $\Gamma^\sigma\backslash X^\sigma$ as a direct sum of complex vector bundles $\Gamma^\sigma\backslash(X^\sigma\times F^{\alpha_j})$ that are all unimodular (note that by hypothesis Γ^σ is torsion-free). The differential form $\theta_\sigma(\mathscr{L}_{\mathbb{C}},h^{\mathscr{L}_{\mathbb{C}}})$ defined in [5, Def. 2.5] therefore vanishes. And, by [5, Th. 0.2], we conclude that

$$\begin{aligned}(8.12.1)\quad & 2[\log RT_\sigma(\mathscr{M},\mathscr{L}_{\mathbb{C}};f,X)-\log T^\sigma_{(\mathscr{M},g)}(\mathscr{L}_{\mathbb{C}})]\\ &= -\frac{1}{4}\sum_{x\in\mathrm{Crit}(f)\cap\mathscr{M}^\sigma}(-1)^{\mathrm{ind}(f|_{\mathscr{M}^\sigma},x)}\sum_j(n_+(\beta_j,x)-n_-(\beta_j,x))\cdot C_j\cdot\mathrm{trace}[\sigma|\mathscr{L}_x],\end{aligned}$$

where:

– the integers $n_+(\beta_j,x)$ and $n_-(\beta_j,x)$ respectively denote the number of positive and negative eigenvalues of the Hessian $d^2f(x)$ acting on $N(\beta_j)_x$, where $N(\beta_j)$ is the subbundle of the normal bundle N to $\mathscr{M}^\sigma$ on which σ acts by $e^{\pm i\beta_j}$;

– the constant C_j is related to the equivariant torsion of a sphere. See [24, §11].

Now, for our particular choice of Morse data and replacing the locally symmetric metric g by the modified metric g', we deduce from (8.7.1) that the expression

$$\sum_j(n_+(\beta_j,x)-n_-(\beta_j,x))\cdot C_j\cdot\mathrm{trace}[\sigma|\mathscr{L}_x]$$

is constant for critical points x in a single connected component $\mathscr{M}_0$ of the fixed point set $\mathscr{M}^\sigma$. Indeed, $n_+(\beta_j,x)=\dim N(\beta_j)|_{\mathscr{M}_0}$, $n_-(\beta_j,x)=0$, and, since σ preserves the flat connection, $\mathrm{trace}(\sigma|\mathscr{L}_x)$ is also constant. Therefore,

$$\begin{aligned}-\frac{1}{4}\sum_{x\in\mathrm{Crit}(f)\cap\mathscr{M}^\sigma}&(-1)^{\mathrm{ind}(f|_{\mathscr{M}^\sigma},x)}\sum_j(n_+(\beta_j,x)-n_-(\beta_j,x))\cdot C_j\cdot\mathrm{trace}[\sigma|\mathscr{L}_x]\\ &=\sum_{\mathscr{M}_0\in\pi_0(\mathscr{M}^\sigma)}\mathrm{constant}(\mathscr{M}_0)\sum_{x\in\mathrm{Crit}(f)\cap\mathscr{M}_0}(-1)^{\mathrm{ind}(f|_{\mathscr{M}_0},x)}\\ &=\sum_{\mathscr{M}_0\in\pi_0(\mathscr{M}^\sigma)}\mathrm{constant}(\mathscr{M}_0)\chi(\mathscr{M}_0)=0.\end{aligned}$$

Thus,

$$\log RT_\sigma(\mathscr{M},\mathscr{L}_{\mathbb{C}};f,X)-\log T^\sigma_{(\mathscr{M},g')}(\mathscr{L}_{\mathbb{C}})=0.$$

To conclude, note that by the anomaly formula of Bismut-Zhang [5, Th. 0.1],(9)

$$T^\sigma_{(\mathscr{M},g')}(\mathscr{L}_{\mathbb{C}})=T^\sigma_{(\mathscr{M},g)}(\mathscr{L}_{\mathbb{C}})$$

when all components of the fixed point set are odd-dimensional. □

(9)Here again, we use that the differential form $\theta_\sigma(\mathscr{L},h^{\mathscr{L}})$ vanishes.

8.13. Growth of torsion in cohomology of locally symmetric spaces. — Under the hypotheses of Section 8.6 — in particular the hypothesis that $\widetilde{\rho}$ is strongly twisted acyclic — it follows from Theorem 4.11 that

$$\frac{\log T^{\sigma}_{\Gamma_n\backslash X}(\rho)}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} \longrightarrow t_X^{(2)\sigma}(\rho). \tag{8.13.1}$$

From now on we will furthermore assume that $\delta(G^{\sigma})=1$; note that this forces X^{σ} to be odd dimensional. Theorem 8.12 therefore implies that

$$\log T^{\sigma}_{\Gamma_n\backslash X}(\rho)=\log RT_{\sigma}(\mathscr{M}_n,\mathscr{L}_{\mathbb{C}};f_n,X_n).$$

Then Theorem 8.10, the remark following it, and Equation (8.13.1) imply the following

8.14. Corollary. — *Under the above hypotheses (in particular with σ of* prime order p*) suppose furthermore that*

$$\log|H^*(\Gamma_n,\mathscr{O})[p^{\infty}]| = o(|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma}))$$

and

$$\log|H^*(\Gamma_n^{\sigma},\mathscr{O}_{\mathbb{F}_p})| = o(\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})).$$

Then

$$\frac{1}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})}\Big[\sum_i(-1)^i\log R^i_{\sigma}(\Gamma_n,\rho) - \sum_i(-1)^i\Big(\log\big|H^i(\Gamma_n,\mathscr{O})[p^{-1}]^{\sigma-1}\big| - \frac{1}{p-1}\log\big|H^i(\Gamma_n,\mathscr{O})[p^{-1}]^{P(\sigma)}\big|\Big)\Big] \xrightarrow{n\to\infty} t_X^{(2)\sigma}(\rho).$$

In particular, if ρ is strongly acyclic, then

$$\frac{-\sum_i(-1)^i\Big(\log\big|H^i(\mathscr{M},\mathscr{L})[p^{-1}]^{\sigma-1}\big| - \frac{1}{p-1}\log\big|H^i(\mathscr{M},\mathscr{L})[p^{-1}]^{P(\sigma)}\big|\Big)}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})} \xrightarrow{n\to\infty} t_X^{(2)\sigma}(\rho).$$

One can deduce from this an unconditional cohomology growth result:

8.15. Corollary. — *Enforce the same notations and hypotheses as in Corollary* 8.14, *with* no a priori cohomology growth assumptions. *Furthermore, assume that ρ is strongly acyclic and that $\mathscr{O}_{\mathbb{F}_p}$ is trivial. Then*

$$\limsup\frac{\sum_i\log|H^i(\Gamma_n,\mathscr{O})_{\mathrm{tors}}|}{\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})}>0.$$

Remark. — The local system $(\Gamma_n\backslash X,\mathscr{O}_{\mathbb{F}_p})$ is trivial if and only if Γ_n is contained in the kernel of $\rho \bmod p$. We can therefore construct many examples using Proposition 4.5 relative to the over group $\Gamma=\ker(\rho \bmod p)$.

Proof. — Suppose that not both of the growth hypotheses of Corollary 8.14 hold.

– If $\limsup\log|H^*(\Gamma_n,\mathscr{O})[p^{\infty}]|/\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})>0$, we are done.
– Suppose the mod p cohomology of $((\Gamma_n\backslash X)^{\sigma},\mathscr{O}_{\mathbb{F}_p})$ is large, i.e.,

$$\frac{\log|H^*((\Gamma_n\backslash X)^{\sigma},\mathscr{O}_{\mathbb{F}_p})|}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^{\sigma}\backslash G^{\sigma})}\not\to 0.$$

Because $\mathscr{O}_{\mathbb{F}_p}|_{(\Gamma_n\backslash X)^\sigma}$ is trivial, the latter implies that

$$\frac{\log|H^*((\Gamma_n\backslash X)^\sigma,\mathbb{F}_p)|}{|H^1(\sigma,\Gamma_n)|\operatorname{vol}(\Gamma_n^\sigma\backslash G^\sigma)}\not\to 0.$$

The conclusion follows by Smith theory [8, §III]. Indeed [8, §III.4.1],

$$\begin{aligned}\dim_{\mathbb{F}_p} H^*((\Gamma_n\backslash X)^\sigma,\mathbb{F}_p) &\leqslant \dim_{\mathbb{F}_p} H^*(\Gamma_n\backslash X,\mathbb{F}_p)\\ &=\frac{1}{\operatorname{rank}\mathscr{O}}\dim_{\mathbb{F}_p} H^*(\Gamma_n\backslash X,\mathscr{O}_{\mathbb{F}_p}),\end{aligned}\tag{8.15.1}$$

where the final equality follows because $\mathscr{O}_{\mathbb{F}_p}$ is trivial. Because ρ is strongly acyclic, $(\Gamma_n\backslash X,\mathscr{O})$ has no rational cohomology. The desired conclusion then follows by the universal coefficient theorem.

Otherwise, both a priori cohomology growth hypotheses from Corollary 8.14 are satisfied. We thus apply Corollary 8.14, whose conclusion is more refined. Note that it follows from Theorems 5.3 and 6.14 (and the computations in the non-twisted case done in [3]) that $t_X^{(2)\sigma}(\rho)\neq 0$ whenever $\delta(G^\sigma)=1$. □

Remark. — Corollaries 8.14 and 8.15 were one major source of inspiration for this paper. We sought to understand when $t_X^{(2)\sigma}(\rho)\neq 0$ in order to detect torsion cohomology growth. Theorem 1.4 of the Introduction follows from Corollary 8.15.

8.16. Comparison with p-adic methods. — Let $\mathbf{G}$ be an algebraic group over $\mathbb{Z}$ which is smooth over $\mathbb{Z}[N^{-1}]$ and for which $\mathbf{G}_{\mathbb{Q}}$ is semisimple. As a byproduct of their study of completed cohomology [10], Calegari and Emerton are able to prove nontrivial upper and lower bounds on cohomology growth for the family of groups $\Gamma_{p^n}=\ker(\mathbf{G}(\mathbb{Z})\to\mathbf{G}(\mathbb{Z}/p^n\mathbb{Z}))$ for any prime $p\nmid N$. Using Poincaré duality for completed cohomology, they show

$$\dim_{\mathbb{F}_p} H^*(\Gamma_{p^n},\mathbb{F}_p)\gg[\Gamma_1:\Gamma_{p^n}]^{1-\alpha},$$

where $\alpha=\dim(G/K)/\dim G$; Calegari and Emerton prove this when $\{\Gamma_{p^n}\}$ is a family of 3-manifold groups in [11] and Calegari extends this to general $\mathbf{G}$ in [9]. For any local system $\mathscr{L}$ arising from a representation of $\mathbf{G}$ defined over $\mathbb{Q}$ with $\mathscr{L}_{\mathbb{Q}}$ acyclic, they deduce that

$$\log|H^*(\Gamma_{p^n},\mathscr{L})_{\mathrm{tors}}|\geqslant\log|H^*(\Gamma_{p^n},\mathscr{L})[p^\infty]|\gg[\Gamma_1:\Gamma_{p^n}]^{1-\alpha}\tag{8.16.1}$$

as an immediate consequence. It is noteworthy that $\alpha=1/2$ if G is a complex Lie group and $1-\alpha\geqslant 1/3$ for arbitrary G. The resulting lower bound obtained by (8.16.1) is of the same quality as that proven in Corollary (8.15) for quadratic base change of groups with $\delta(G^\sigma)=1$ and is always strictly larger for cyclic base change of degree greater than two for groups with $\delta(G^\sigma)=1$. Nonetheless, these lower bounds do not subsume our main theorems on torsion cohomology growth.

– Suppose $\delta(G^\sigma)=1$. By Corollary 8.15, any family of groups satisfying the hypotheses of Theorem 4.11 exhibits torsion cohomology growth. As shown in Proposition 4.5, fairly general families $\{\Gamma'_q\}$ of congruence subgroups which grow horizontally, e.g. as q varies through a sequence of primes, satisfy these hypotheses. However, (8.16.1) does not give any information concerning cohomology growth for such families of horizontally growing congruence subgroups.

– Suppose $\delta(G^\sigma) = 1$ and consider the family of congruence subgroups $\{\Gamma_{p^n}\}$ of full level p^n. Both Corollary 8.14 and (8.16.1) yield lower bounds on torsion cohomology growth. However, their origins should be regarded as very distinct.

Cohomology classes accounted for by (8.16.1) are an aggregate of $\bmod p$ congruences between $(\bmod p)$ automorphic representations of $\mathbf{G}$ of arbitrary level.

On the other hand, the cohomology classes accounted for by Corollary 8.14 conjecturally arise by base change transfer over $\mathbb{Z}$ [12, 22]. Partial evidence for this transfer occurs in Theorem 6.14, which may be regarded as 'numerical base change transfer over $\mathbb{Z}$ at infinite level' (see [22] for some special cases of numerical base change transfer over $\mathbb{Z}$ at finite level).

Base change for torsion cohomology leads us to expect that torsion witnessed in Corollary 8.14 is supported at the same primes as torsion in the cohomology of locally symmetric spaces for G^σ; computations suggest that the latter primes are large and irregular [31]. On the other hand, torsion witnessed through (8.16.1) is supported at a single prime p and gives no information about the prime-to-p part of torsion cohomology.

References

[1] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault & I. Samet – "On the growth of L^2-invariants for sequences of lattices in Lie groups", `arXiv:1210.2961`, 2012.

[2] D. Barbasch & H. Moscovici – "L^2-index and the Selberg trace formula", *J. Funct. Anal.* **53** (1983), no. 2, p. 151–201.

[3] N. Bergeron & A. Venkatesh – "The asymptotic growth of torsion homology for arithmetic groups", *J. Inst. Math. Jussieu* **12** (2013), no. 2, p. 391–447.

[4] J.-M. Bismut & W. Zhang – *An extension of a theorem by Cheeger and Müller*, Astérisque, vol. 205, Société Mathématique de France, Paris, 1992, With an appendix by François Laudenbach.

[5] ———, "Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle", *Geom. Funct. Anal.* **4** (1994), no. 2, p. 136–212.

[6] A. Borel, J.-P. Labesse & J. Schwermer – "On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields", *Compositio Math.* **102** (1996), no. 1, p. 1–40.

[7] A. Bouaziz – "Formule d'inversion d'intégrales orbitales tordues", *Compositio Math.* **81** (1992), no. 3, p. 261–290.

[8] G. E. Bredon – *Introduction to compact transformation groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York-London, 1972.

[9] F. Calegari – "Blog post: torsion in the cohomology of co-compact arithmetic lattices", `http://galoisrepresentations.wordpress.com/2013/02/06/`.

[10] F. Calegari & M. Emerton – "Bounds for multiplicities of unitary representations of cohomological type in spaces of cusp forms", *Ann. of Math. (2)* **170** (2009), no. 3, p. 1437–1446.

[11] ———, "Mod-p cohomology growth in p-adic analytic towers of 3-manifolds", *Groups Geom. Dyn.* **5** (2011), no. 2, p. 355–366.

[12] F. Calegari & A. Venkatesh – "A torsion Jacquet–Langlands correspondence", `arXiv:1212.3847`, 2012.

[13] L. Clozel – "Changement de base pour les représentations tempérées des groupes réductifs réels", *Ann. Sci. École Norm. Sup. (4)* **15** (1982), no. 1, p. 45–115.

[14] P. Delorme – "Théorème de Paley-Wiener invariant tordu pour le changement de base $\mathbf{C}/\mathbf{R}$", *Compositio Math.* **80** (1991), no. 2, p. 197–228.

[15] Harish-Chandra – "Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula", *Ann. of Math. (2)* **104** (1976), no. 1, p. 117–201.
[16] R. A. Herb & J. A. Wolf – "The Plancherel theorem for general semisimple groups", *Compositio Math.* **57** (1986), no. 3, p. 271–355.
[17] S. Illman – "Smooth equivariant triangulations of G-manifolds for G a finite group", *Math. Ann.* **233** (1978), no. 3, p. 199–220.
[18] F. F. Knudsen & D. Mumford – "The projectivity of the moduli space of stable curves. I. Preliminaries on 'det' and 'Div'", *Math. Scand.* **39** (1976), no. 1, p. 19–55.
[19] J.-P. Labesse – "Pseudo-coefficients très cuspidaux et K-théorie", *Math. Ann.* **291** (1991), no. 4, p. 607–616.
[20] J.-P. Labesse & J.-L. Waldspurger – *La formule des traces tordue d'après le Friday Morning Seminar*, CRM Monograph Series, vol. 31, American Mathematical Society, Providence, RI, 2013.
[21] R. P. Langlands – *Base change for* GL(2), Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J., 1980.
[22] M. Lipnowski – "Equivariant torsion and base change", *Algebra Number Theory* **9** (2015), no. 10, p. 2197–2240.
[23] ———, "The equivariant Cheeger–Müller theorem on locally symmetric spaces", *J. Inst. Math. Jussieu* **15** (2016), no. 1, p. 165–202, See also `arXiv:1312.2543v2`; version 2 has corrections that we refer to not appearing in the published JIMJ version.
[24] J. Lott & M. Rothenberg – "Analytic torsion for group actions", *J. Differential Geom.* **34** (1991), no. 2, p. 431–481.
[25] W. Lück – "Analytic and topological torsion for manifolds with boundary and symmetry", *J. Differential Geom.* **37** (1993), no. 2, p. 263–322.
[26] W. Müller – "Analytic torsion and R-torsion for unimodular representations", *J. Amer. Math. Soc.* **6** (1993), no. 3, p. 721–753.
[27] W. Müller & J. Pfaff – "On the asymptotics of the Ray-Singer analytic torsion for compact hyperbolic manifolds", *Internat. Math. Res. Notices* (2013), no. 13, p. 2945–2983.
[28] M. Olbrich – "L^2-invariants of locally symmetric spaces", *Doc. Math.* **7** (2002), p. 219–237.
[29] J. Rohlfs – "Lefschetz numbers for arithmetic groups", in *Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989)*, Lect. Notes in Math., vol. 1447, Springer, Berlin, 1990, p. 303–313.
[30] J. Rohlfs & B. Speh – "Lefschetz numbers and twisted stabilized orbital integrals", *Math. Ann.* **296** (1993), no. 2, p. 191–214.
[31] M. H. Şengün – "On the integral cohomology of Bianchi groups", *Experiment. Math.* **20** (2011), no. 4, p. 487–505.
[32] J.-P. Serre – *Cohomologie galoisienne*, fifth ed., Lect. Notes in Math., vol. 5, Springer-Verlag, Berlin, 1994.

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