Indranil Biswas & Oscar García-Prada
Anti-holomorphic involutions of the moduli spaces of Higgs bundles

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ANTI-HOLOMORPHIC INVOLUTIONS OF
THE MODULI SPACES OF HIGGS BUNDLES

BY INDRANIL BISWAS & OSCAR GARCÍA-PRADA

Abstract. — We study anti-holomorphic involutions of the moduli space of $G$-Higgs bundles over a compact Riemann surface $X$, where $G$ is a complex semisimple Lie group. These involutions are defined by fixing anti-holomorphic involutions on both $X$ and $G$. We analyze the fixed point locus in the moduli space and their relation with representations of the orbifold fundamental group of $X$ equipped with the anti-holomorphic involution. We also study the relation with branes. This generalizes work by Biswas–García-Prada–Hurtubise and Baraglia–Schaposnik.

Résumé (Invocations anti-holomorphes des espaces de modules de fibrés de Higgs)

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Keywords. — Higgs $G$-bundle, reality condition, branes, character variety.

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1. Introduction

Let $G$ be a complex semisimple affine algebraic group with Lie algebra $\mathfrak{g}$. Let $X$ be a compact connected Riemann surface. A $G$–Higgs bundle over $X$ is a pair $(E, \varphi)$, where $E$ is a holomorphic principal $G$-bundle over $X$ and $\varphi$ is a holomorphic section of $E(\mathfrak{g}) \otimes K$ with $E(\mathfrak{g})$ being the vector bundle associated to $E$ for the adjoint action of $G$ on $\mathfrak{g}$ and $K$ being the canonical line bundle on $X$. We consider the moduli space of polystable $G$-Higgs bundles $\mathcal{M}(G)$. This has the structure of a hyper-Kähler manifold outside the singular locus.

Let $\alpha : X \to X$ and $\sigma : G \to G$ be anti-holomorphic involutions. We define the two involutions (see Section 4.1 for details)

$$\iota(\alpha, \sigma)^\pm : \mathcal{M}(G) \to \mathcal{M}(G)$$

$$(E, \varphi) \mapsto (\alpha^* \sigma(E), \pm \alpha^* \sigma(\varphi)).$$

(1.1)

The goal of this paper is to describe the fixed points of these involutions. The fixed points are given by the image of moduli spaces of $G$-Higgs bundles satisfying a reality condition determined by $\alpha$ and $\sigma$, and an element $c \in Z_2^\sigma$, where $Z$ is the center of $G$ and $Z_2^\sigma$ is the group of elements of order two in $Z$ fixed by $\sigma$. For the involution $\iota(\alpha, \sigma)^+$, these are the moduli space of pseudo-real Higgs bundles considered in [5], to which we refer here as $\iota(\alpha, \sigma)^+$, but the different sign on $\varphi$ gives a different reality condition on the moduli space of Higgs bundles, defining objects that we call $(\alpha, \sigma, c, +)$-pseudo-real $G$-Higgs bundles. When the element $c \in Z_2^\sigma$ is trivial we call these objects real $G$-Higgs bundles.

The involution $\iota(\alpha, \sigma)^-$ is studied by Baraglia-Schaposnik [3] when $\sigma$ is the anti-holomorphic involution $\tau$ corresponding to a compact real form of $G$ (see also [19]). In [4], they consider the involutions $\iota(\alpha, \sigma)^+$ obtained as a result of composing $\iota(\alpha, \tau)^-$ with the holomorphic involution $\iota^-(\theta)$ of $\mathcal{M}(G)$ defined by $\iota^-(E, \varphi) = (\theta(E), -\theta(\varphi))$, where $\theta$ is the holomorphic involution of $G$ given by $\theta = \sigma\tau$ (here one takes a compact conjugation $\tau$ commuting with $\sigma$). In fact, if we consider the involutions

$$\iota(\theta)^\pm : \mathcal{M}(G) \to \mathcal{M}(G)$$

$$(E, \varphi) \mapsto (\theta(E), \pm \theta(\varphi)),$$

(1.2)

one has

$$\iota(\alpha, \sigma)^\pm = \iota^+(\theta) \circ \iota^-(\alpha, \tau).$$

The involutions (1.2) have been studied in [12, 13, 15].

In the language of branes [20], the fixed points of $\iota(\alpha, \sigma)^+$ are $(A, A, B)$–branes, while the fixed points of $\iota(\alpha, \sigma)^-$ are $(A, B, A)$–branes. What these mean is that the fixed points of $\iota(\alpha, \sigma)^-$ are complex Lagrangian submanifolds with respect to the complex structure $J_2$ defined on $\mathcal{M}(G)$ by the complex structure of $G$, while the fixed points of $\iota(\alpha, \sigma)^+$ are complex Lagrangian submanifolds with respect to the complex structure $J_3 = J_1J_2$ obtained by combining $J_2$ with the natural complex structure $J_1$.
defined on the moduli space of Higgs bundles for the Riemann surface \( X \). The study of these branes is of great interest in connection with mirror symmetry and the Langlands correspondence in the theory of Higgs bundles (see [20, 18, 3, 2]).

We then identify these involutions in the moduli space of representations of the fundamental group of \( X \) in \( G \), and describe the fixed points corresponding to the \((\alpha, \sigma, c, \pm)\)-pseudo-real \( G \)-Higgs bundles in terms of representations of the orbifold fundamental group of \((X, \alpha)\) in a group whose underlying set is \( G \times \mathbb{Z}/2\mathbb{Z} \). The group structure on \( G \times \mathbb{Z}/2\mathbb{Z} \) is constructed using the element \( c \in \mathbb{Z}^2 \) and an action of \( \mathbb{Z}/2\mathbb{Z} \) on \( G \) which depends on the sign of the pseudo-reality condition; more precisely, this action is given by the conjugation \( \sigma \) in the “+” case, and the action of \( \theta = \sigma \tau \) in the “−” case, where \( \tau \) is a compact conjugation commuting with \( \sigma \). When \( c \) is trivial we obtain the semi-direct products of \( G \) with \( \mathbb{Z}/2\mathbb{Z} \) for the action \( \sigma \).

The results of this paper have a straightforward generalization to the case in which \( G \) is reductive. In this situation the fundamental group of \( X \) is replaced by its universal central extension.

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2. \( G \)-Higgs bundles and representations of the fundamental group

2.1. Moduli space of \( G \)-Higgs bundles. — Let \( G \) be a complex semisimple affine algebraic group. Its Lie algebra will be denoted by \( \mathfrak{g} \). Let \( X \) be an irreducible smooth projective curve defined over \( \mathbb{C} \), equivalently, it is a compact connected Riemann surface. Let \( g_x \) be the genus of \( X \); throughout we assume that \( g_x \geq 2 \). The canonical line bundle of \( X \) will be denoted by \( K \). For a principal \( G \)-bundle \( E \), let \( E(\mathfrak{g}) := E \times^G \mathfrak{g} \) be the adjoint vector bundle for \( E \).

A \( G \)-Higgs bundle over \( X \) is a pair \((E, \varphi)\), where \( E \) is a holomorphic principal \( G \)-bundle over \( X \) and \( \varphi \) is a holomorphic section of \( E(\mathfrak{g}) \otimes K \). Two \( G \)-Higgs bundles \((E, \varphi)\) and \((F, \psi)\) are isomorphic if there is a holomorphic isomorphism \( f: E \to F \) such that the induced isomorphism

\[
\text{Ad}(f) \otimes \text{Id}_K : E(\mathfrak{g}) \otimes K \longrightarrow F(\mathfrak{g}) \otimes K
\]

sends \( \varphi \) to \( \psi \).

There are notions of (semi)stability and polystability for \( G \)-Higgs bundles (see [8, 14, 7] for example). A \( G \)-Higgs bundle \((E, \varphi)\) is said to be stable (respectively, semistable) if for every parabolic subgroup \( P \subset G \), every holomorphic reduction \( \sigma: E_P \to E \) to \( P \) such that

\[
\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K) \subset H^0(X, E(\mathfrak{g}) \otimes K)
\]
and every strictly antidominant character $\chi$ of $P$, we have that $\deg E_P(\chi) > 0$ (respectively, $\deg E_P(\chi) \geq 0$). A Higgs bundle $(E, \varphi)$ is polystable if it is semistable and for every $P$, every reduction and every $\chi$ as above such that $\deg E_P(\chi) = 0$, there is a holomorphic reduction $E_L \subset E$ to a Levi subgroup $L \subset P$ such that $\varphi \in H^0(X, E_L(L) \otimes K)$.

Let $\mathcal{M}(G)$ denote the moduli space of semistable $G$-Higgs bundles of fixed topological type. This moduli space has the structure of a complex normal quasiprojective variety of dimension $\dim G(g_X - 1)$.

2.2. $G$-Higgs bundles and Hitchin equations. — As above, let $G$ be a complex semisimple affine algebraic group. Let $H \subset G$ be a maximal compact subgroup. Let $(E, \varphi)$ be a $G$-Higgs bundle over a compact Riemann surface $X$. By a slight abuse of notation, we shall denote the $C^\infty$-objects underlying $E$ and $\varphi$ by the same symbols. In particular, the Higgs field can be viewed as a $(1, 0)$-form $\varphi \in \Omega^{1, 0}(E(g))$ with values in $E(g)$. Let

$\tau : \Omega^{1, 0}(E(g)) \rightarrow \Omega^{0, 1}(E(g))$

be the isomorphism induced by the compact conjugation of $g$ (with respect to $H$) combined with the complex conjugation on complex 1-forms. Given a $C^\infty$ reduction of structure group $h$ of the principal $G$-bundle $E$ to $H$, we denote by $F_h$ the curvature of the unique connection compatible with $h$ and the holomorphic structure on $E$; see [1, pp.191–192, Prop. 5] for the connection.

Theorem 2.1. — There is a reduction $h$ of structure group of $E$ from $G$ to $H$ that satisfies the Hitchin equation

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if $(E, \varphi)$ is polystable.

Theorem 2.1 was proved by Hitchin [17] for $G = SL(2, \mathbb{C})$, and in [23, 24, 7] for the general case.

Remark 2.2. — When $G$ is reductive the equation in Theorem 2.1 is replaced by the equation

$$F_h - [\varphi, \tau(\varphi)] = c \omega,$$

where $\omega$ is a Kähler form on $X$ and $c$ is an element in the center of the Lie algebra of $G$, which is determined by the topology of $E$.

From the point of view of moduli spaces it is convenient to fix a $C^\infty$ principal $H$-bundle $E_H$ and study the moduli space of solutions to Hitchin’s equations for a pair $(A, \varphi)$ consisting of a $H$-connection $A$ on $E_H$ and a section $\varphi \in \Omega^{1, 0}(X, E_H(g))$:

$$F_A - [\varphi, \tau(\varphi)] = 0 \quad \overline{\partial}_A \varphi = 0.$$  \hspace{1cm} (2.1)

Here $d_A$ is the covariant derivative associated to $A$, and $\overline{\partial}_A$ is the $(0, 1)$ part of $d_A$. The $(0, 1)$ part of $d_A$ defines a holomorphic structure on $E_H$. The gauge group $\mathscr{H}$
of $E_H$ acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}^{\text{gauge}}(G) := \{(A, \varphi) \text{ satisfying (2.1)}\}/\mathcal{G}.$$ 

Now, Theorem 2.1 can be reformulated as follows.

**Theorem 2.3.** — There is a homeomorphism

$$\mathcal{M}(G) \cong \mathcal{M}^{\text{gauge}}(G).$$

To explain this correspondence we interpret the moduli space of $G$-Higgs bundles in terms of pairs $(\mathcal{D}_E, \varphi)$ consisting of a $\mathcal{D}$-operator (holomorphic structure) $\mathcal{D}_E$ on the $C^\infty$ principal $G$-bundle $E_G$ obtained from $E_H$ by the extension of structure group $H \to G$, and $\varphi \in \Omega^{1,0}(X, E_G(\mathfrak{g}))$ satisfying $\mathcal{D}_E \varphi = 0$. Such pairs are in one-to-one correspondence with $G$-Higgs bundles $(E, \varphi)$, where $E$ is the holomorphic $G$-bundle defined by the operator $\mathcal{D}_E$ on $E_G$. The equation $\mathcal{D}_E \varphi = 0$ is equivalent to the condition that $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$. The moduli space of polystable $G$-Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{ (\mathcal{D}_E, \varphi) \mid \mathcal{D}_E \varphi = 0 \text{ and the $G$-Higgs bundle is polystable} \}/\mathcal{G},$$

where $\mathcal{G}$ is the gauge group of $E_G$, which is in fact the complexification of $\mathcal{H}$. Since there is a one-to-one correspondence between $H$-connections on $E_H$ and $\mathcal{D}$-operators on $E_G$, the correspondence given in Theorem 2.3 can be reformulated by saying that in the $\mathcal{G}$-orbit of a polystable $G$-Higgs bundle $(\mathcal{D}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\mathcal{D}_E, \varphi)$ whose corresponding pair $(d_A, \varphi)$ satisfies the Hitchin equation $F_A - \lbrack \varphi, \gamma(\varphi) \rbrack = 0$ with this pair $(d_A, \varphi)$ being unique up to $H$-gauge transformations.

2.3. Higgs bundles and representations. — Fix a base point $x_0 \in X$. By a representation of $\pi_1(X, x_0)$ in $G$ we mean a homomorphism $\pi_1(X, x_0) \to G$. After fixing a presentation of $\pi_1(X, x_0)$, the set of all such homomorphisms, $\text{Hom}(\pi_1(X, x_0), G)$, can be identified with the subset of $G^{2g_X}$ consisting of $2g_X$-tuples $(A_1, B_1, \ldots, A_{g_X}, B_{g_X})$ satisfying the algebraic equation $\prod_{i=1}^{2g_X} |A_i, B_i| = 1$. This shows that $\text{Hom}(\pi_1(X, x_0), G)$ is a complex algebraic variety.

The group $G$ acts on $\text{Hom}(\pi_1(X, x_0), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g \rho(\gamma) g^{-1},$$

where $g \in G$, $\rho \in \text{Hom}(\pi_1(X, x_0), G)$ and $\gamma \in \pi_1(X, x_0)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X, x_0), G)$ consisting of reductive representations, the orbit space is Hausdorff. We recall that a reductive representation is one whose composition with the adjoint representation in $\mathfrak{g}$ decomposes as a direct sum of irreducible representations. This is equivalent to the condition that the Zariski closure of the image of $\pi_1(X, x_0)$ in $G$ is a reductive group. Define the moduli space of representations of $\pi_1(X, x_0)$ in $G$ to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G.$$
For another point $x' \in X$, the fundamental groups $\pi_1(X,x_0)$ and $\pi_1(X,x')$ are identified by an isomorphism unique up to an inner automorphism. Consequently, $\mathcal{R}(G)$ is independent of the choice of the base point $x_0$.

One has the following (see e.g. [16], [25]).

**Theorem 2.4.** — The moduli space $\mathcal{R}(G)$ has the structure of a normal complex variety. Its smooth locus is equipped with a holomorphic symplectic form.

Given a representation $\rho: \pi_1(X,x_0) \to G$, there is an associated flat principal $G$-bundle on $X$, defined as $E_\rho = \tilde{X} \times^\rho G$,

where $\tilde{X} \to X$ is the universal cover associated to $x_0$ and $\pi_1(X,x_0)$ acts on $G$ via $\rho$.

This gives in fact an identification between the set of equivalence classes of representations $\text{Hom}(\pi_1(X),G)/G$ and the set of equivalence classes of flat principal $G$-bundles, which in turn is parametrized by the (nonabelian) cohomology set $H^1(X,G)$.

We have the following:

**Theorem 2.5.** — There is a homeomorphism $\mathcal{R}(G) \cong \mathcal{M}(G)$.

The moduli spaces $\mathcal{M}(G)$ and $\mathcal{R}(G)$ are sometimes referred as the Dolbeault and Betti moduli spaces, respectively.

The proof of Theorem 2.5 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let $E_H$ be, as above, a $C^\infty$ principal $G$-bundle over $X$ and $E_H$ a $C^\infty$ reduction of structure group of it to $H$. Every $G$-connection $D$ on $E_G$ decomposes uniquely as $D = d_A + \psi$,

where $d_A$ is an $H$-connection on $E_H$ and $\psi \in \Omega^1(X, E_H(\sqrt{-1}h))$. Let $F_A$ be the curvature of $d_A$. We consider the following set of equations for the pair $(d_A, \psi)$:

\begin{align}
F_A + \frac{i}{2}[\psi, \psi] &= 0 \\
d_A\psi &= 0 \\
d^*_A\psi &= 0.
\end{align}

These equations are invariant under the action of $\mathcal{H}$, the gauge group of $E_H$. A theorem of Corlette [10], and Donaldson [11] for $G = \text{SL}(2,\mathbb{C})$, says the following.

**Theorem 2.6.** — There is a homeomorphism between

$$\{\text{Reductive } G\text{-connections } D \mid F_D = 0\}/\mathcal{G}$$

and

$$\{(d_A, \psi) \text{ satisfying (2.2)}\}/\mathcal{H}.$$
In fact, the holomorphic symplectic structure on $E$ define homeomorphism action of $\tilde{g} \in \mathcal{G}$ on $h_0$. The equation $d_A^* \psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$-equivariant map $\tilde{X} \to G/H$ corresponding to the new reduction of structure group $h$.

To complete the argument, leading to Theorem 2.5, we just need Theorem 2.1 and the following simple result.

**Proposition 2.7.** — The correspondence $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ defines a homeomorphism

$\{ (d_A, \varphi) \text{ satisfying (2.1)} \} / \mathcal{H} \cong \{ (d_A, \psi) \text{ satisfying (2.2)} \} / \mathcal{H}$.

**2.4. The moduli space as a hyper-Kähler quotient.** — We will see now that the moduli space $\mathcal{M}(G)$ has a hyper-Kähler structure. For this, recall first that a hyper-Kähler manifold is a differentiable manifold $M$ equipped with a Riemannian metric $g$ and complex structures $J_i$, $i = 1, 2, 3$ satisfying the quaternion relations $J_i^2 = -I$, $J_i J_j = -J_j J_i$, $J_3 = -J_1 J_2 = J_2 J_1$, $J_2 = -J_1 J_3 = J_1 J_3$, and $J_1 J_3 = J_1 J_2$ such that if we define $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, then $(g, J_i, \omega_i)$ is a Kähler structure on $M$. Let $\Omega_i$ denote the holomorphic symplectic structure on $\mathcal{M}(G)$ with respect to the complex structure $J_i$.

In fact, $\Omega_1 = \omega_2 + \sqrt{-1} \omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1} \omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1} \omega_2$.

One way to understand the non-abelian Hodge theory correspondence mentioned above is through the analysis of the hyper-Kähler structure of the moduli spaces involved. We explain how these can be obtained as hyper-Kähler quotients. For this, let $E_G$ be a smooth principal $G$-bundle over $X$, and let $E_H$ be a fixed reduction of $E_G$ to the maximal compact subgroup $H$. The set $\mathcal{A}$ of $H$-connections on $E_H$ is an affine space modelled on $\Omega^1(X, E_G(h))$. Via the Chern correspondence, $\mathcal{A}$ is in one-to-one correspondence with the set $\mathcal{C}$ of holomorphic structures on $E_G$ [1, pp. 191–192, Prop. 5], which is an affine space modelled on $\Omega^{0,1}(X, E_G(g))$. Let us denote $\Omega = \Omega^{1,0}(X, E_G(g))$. We consider $\mathcal{X} = \mathcal{A} \times \Omega$. Via the identification $\mathcal{A} \cong \mathcal{C}$, we have for $\alpha \in \Omega^{0,1}(X, E_G(g))$ and $\psi \in \Omega^{1,0}(X, E_G(g))$ the following three complex structures on $\mathcal{X}$:

$J_1(\alpha, \psi) = (\sqrt{-1} \alpha, \sqrt{-1} \psi)$

$J_2(\alpha, \psi) = (-\sqrt{-1} \tau(\psi), \sqrt{-1} \tau(\alpha))$

$J_3(\alpha, \psi) = (\tau(\psi), -\tau(\alpha)),$

where $\tau$ is the conjugation on $g$ defining its compact form $\mathfrak{h}$ (determined fiber-wise by the reduction to $E_H$), combined with the complex conjugation on complex 1-forms.

One has also a Riemannian metric $g$ defined on $\mathcal{X}$: for $\alpha \in \Omega^{0,1}(X, E_G(g))$ and $\psi \in \Omega^{1,0}(X, E_G(g))$,

$$g((\alpha, \psi), (\alpha, \psi)) = -2\sqrt{-1} \int_X B(\tau(\alpha), \alpha) + B(\psi, \tau(\psi)),$$

where $B$ is the Killing form.

Clearly, $J_i$, $i = 1, 2, 3$, satisfy the quaternion relations, and define a hyper-Kähler structure on $\mathcal{X}$, with Kähler forms $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, $i = 1, 2, 3$. As shown in [17],
the action of the gauge group $\mathcal{H}$ on $\mathcal{X}$ preserves the hyper-Kähler structure and there are moment maps given by

$$
\mu_1(A, \varphi) = F_A - [\varphi, \tau(\varphi)], \quad \mu_2(A, \varphi) = \text{Re}(\overline{\partial}_E \varphi), \quad \mu_3(A, \varphi) = \text{Im}(\partial_E \varphi).
$$

We have that $\mu^{-1}(0)/\mathcal{H}$, where $\mu = (\mu_1, \mu_2, \mu_3)$ is the moduli space of solutions to the Hitchin equations (2.1). In particular, if we consider the irreducible solutions (equivalently, smooth) $\mu^{-1}(0)$ we have that

$$
\mu^{-1}(0)/\mathcal{H}
$$

is a hyper-Kähler manifold which, by Theorem 2.3, is homeomorphic to the subvariety of smooth points in moduli space $\mathcal{M}(G)$ of stable $G$-Higgs bundles with the topological class of $E_G$.

Let us now see how the moduli of harmonic flat connections on $E_H$ can be realized as a hyper-Kähler quotient. Let $\mathcal{D}$ be the set of $G$-connections on $E_G$. This is an affine space modelled on $\Omega^1(X, E_G(g)) = \Omega^0(X, T^*X \otimes_R E_G(g))$. The space $\mathcal{D}$ has a complex structure $I_1 = 1 \otimes \sqrt{-1}$, which comes from the complex structure of the bundle. Using the complex structure of $X$ we have also the complex structure $I_2 = \sqrt{-1} \otimes \tau$. We can finally consider the complex structure $I_3 = I_1 I_2$.

The reduction to $H$ of the $G$-bundle $E_G$ together with a Riemannian metric in the conformal class of $X$ defines a flat Riemannian metric $g_D$ on $\mathcal{D}$ which is Kähler for the above three complex structures. Hence $(\mathcal{D}, g_D, I_1, I_2, I_3)$ is also a hyper-Kähler manifold. As in the previous case, the action of the gauge group $\mathcal{H}$ on $\mathcal{D}$ preserves the hyper-Kähler structure and there are moment maps

$$
\mu_1(D) = d_A \psi, \quad \mu_2(D) = \text{Im}(F_D), \quad \mu_3(D) = \text{Re}(F_D),
$$

where $D = d_A + \psi$ is the decomposition of $D$ defined by

$$
E_G(g) = E_H(h) \oplus E_H(\sqrt{-1}h).
$$

Thus the moduli space of solutions to the harmonicity equations (2.2) is the hyper-Kähler quotient defined by

$$
\mu^{-1}(0)/\mathcal{H},
$$

where $\mu = (\mu_1, \mu_2, \mu_3)$. The homeomorphism between the moduli spaces of solutions to the Hitchin and the harmonicity equations is induced from the affine map

$$
\mathcal{A} \times \Omega \longrightarrow \mathcal{D}
$$

$$(d_A, \varphi) \longmapsto d_A + \varphi - \tau(\varphi).$$

One can see easily, for example, that this map sends $\mathcal{A} \times \Omega$ with complex structure $J_2$ to $\mathcal{D}$ with complex structure $I_1$ (see [17]).

Now, Theorems 2.3 and 2.6 can be regarded as existence theorems, establishing the non-emptiness of the hyper-Kähler quotient, obtained by focusing on different complex structures. For Theorem 2.3 one gives a special status to the complex structure $J_1$. Combining the symplectic forms determined by $J_2$ and $J_3$ one has the $J_1$-holomorphic symplectic form $\omega_c = \omega_2 + \sqrt{-1} \omega_3$ on $\mathcal{A} \times \Omega$. The gauge group $\mathcal{G} = \mathcal{H}^C$ acts on $\mathcal{A} \times \Omega$ preserving $\omega_c$. The symplectic quotient construction can also be extended.
to the holomorphic situation (see e.g. [21]) to obtain the holomorphic symplectic quotient \( \{ (\bar{\partial} E, \varphi) \mid \bar{\partial} E \varphi = 0 \} / G \). What Theorem 2.3 says is that for a class \( [(\bar{\partial} E, \varphi)] \) in this quotient to have a representative (unique up to \( H \)-gauge) satisfying \( \mu_1 = 0 \) it is necessary and sufficient that the pair \( (\bar{\partial} E, \varphi) \) be polystable. This identifies the hyper-Kähler quotient to the set of equivalence classes of polystable \( G \)-Higgs bundles on \( E_G \). If one now takes \( J_2 \) on \( \mathcal{A} \times \Omega \) or equivalently \( \mathcal{D} \) with \( I_1 \) and argues in a similar way, one gets Theorem 2.6 identifying the hyper-Kähler quotient to the set of equivalence classes of reductive flat connections on \( E_G \).

3. Real \( G \)-Higgs bundles

3.1. Involutions and conjugations of complex Lie groups. — Let \( G \) be a Lie group. We define

\[
\text{Int}(G) := \{ f \in \text{Aut}(G) \mid f(h) = ghg^{-1}, \text{ for every } h \in G \}.
\]

We have that \( \text{Int}(G) = \text{Ad}(G) \).

We define the group of outer automorphisms of \( G \) as

\[
\text{Out}(G) := \text{Aut}(G) / \text{Int}(G).
\]

We have a sequence

\[
1 \rightarrow \text{Int}(G) \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1.
\]

It is well-known that if \( G \) is a connected complex reductive group then the extension (3.1) splits (see [22]).

Let \( G \) be a complex Lie group and let \( G_\mathbb{R} \) be the underlying real Lie group. We will say that a real Lie subgroup \( G_0 \subset G_\mathbb{R} \) is a real form of \( G \) if it is the fixed point set of a conjugation (anti-holomorphic involution) \( \sigma \) of \( G \).

Now, let \( G \) be simple. A compact real form always exists. This follows from the fact that for a simple group there is a maximal compact subgroup \( U \subset G \), such that \( U^\mathbb{C} = G \). From this we can define a conjugation \( \tau : G \rightarrow G \) such that \( G^\tau = U \). Let \( \text{Conj}(G) \) be the set of conjugations (i.e., anti-holomorphic involutions) of \( G \). We can define the following equivalence relations in \( \text{Conj}(G) \):

\[
\sigma \sim \sigma' \text{ if there is } \alpha \in \text{Int}(G) \text{ such that } \sigma' = \alpha \sigma \alpha^{-1},
\]

We can define a similar relation \( \sim \) in the set \( \text{Aut}_2(G) \) of automorphisms of \( G \) of order 2.

Remark 3.1. — The equivalence relation \( \sim \) for elements in \( \text{Aut}_2(G) \) should not be confused with the inner equivalence, meaning the equivalence relation where two elements are equivalent if they map to the same element in \( \text{Out}(G) \). It is easy to show that if \( \theta \sim \theta' \) then they are inner equivalent.

Cartan [9] shows that there is a bijection

\[
\text{Conj}(G) / \sim \leftrightarrow \text{Aut}_2(G) / \sim.
\]
More concretely, one has that given the compact conjugation \( \tau \), in each class \( \text{Conj}(G)/\sim \) one can find a representative \( \sigma \) commuting with \( \tau \) so that \( \theta := \sigma \tau \) is an element of \( \text{Aut}_2(G) \), and similarly if we start with a class in \( \text{Aut}_2(G)/\sim \).

3.2. Pseudo-real principal \( G \)-bundles. — We use the notation of Section 3.1. Let \( G \) be a semisimple complex affine algebraic group. Let \( \tau \in \text{Conj}(G) \) be a compact conjugation of \( G \), and let \( \sigma \in \text{Conj}(G) \) commuting with \( \tau \), and \( \theta = \sigma \tau \in \text{Aut}_2(G) \).

Let \( Z^\sigma \subset Z \) be the fixed point locus in the center \( Z \subset G \). The subgroup of \( Z^\sigma \) generated by its elements of order two will be denoted by \( Z_2^\sigma \).

Let \( X \) be a compact connected Riemann surface, of genus \( g \geq 2 \), equipped with an anti-holomorphic involution \( \alpha : X \to X \).

**Definition 3.2.** — Let \( E \) be a holomorphic principal \( G \)-bundle over \( X \). Take any \( c \in Z_2^\sigma \). We say that \( E \) is \((\alpha, \sigma, c)\)-pseudo-real if \( E \) is equipped with an anti-holomorphic map \( \tilde{\alpha} : E \to E \) covering \( \alpha \) such that

- \( \tilde{\alpha}(eg) = \tilde{\alpha}(e)\sigma(g) \), for \( e \in E \) and \( g \in G \).
- \( \tilde{\alpha}^2(e) = ec \).

If \( c = 1 \), we say that \( E \) is \((\alpha, \sigma)\)-real.

**Remark 3.3.** — An alternative definition of pseudo-real bundle allows for \( c \) to be any element of \( Z \). However we can modify \( \tilde{\alpha} \) by the action of an element \( a \in Z \) defining a covering map \( \tilde{\alpha}' := \tilde{\alpha}a \). By this, the element \( c \) gets modified by \( c' = a\sigma(a)c \). In particular we can take \( a \) lying in \( Z^\sigma \) and the composition is modified by \( a^2 \). Therefore if \( c \) lies in \((Z^\sigma)^2 \), or more generally is of the form \( \sigma(a)a \) we can normalize our pseudo-real structure to a real one. But since the natural homomorphism \( Z_2^\sigma \to Z^\sigma/(Z^\sigma)^2 \) is surjective we can always assume that \( c \) is of order 2, as we have done in our definition.

**Remark 3.4.** — Sometimes to emphasize the pseudo-real structure we will write \((E, \varphi, \tilde{\alpha})\) for a \( G \)-Higgs bundle \((E, \varphi)\) equipped with a pseudo-real structure \( \tilde{\alpha} \).

Define the quotient \( G_c := G/\langle c \rangle \).

Note that \( \langle c \rangle = \mathbb{Z}/2\mathbb{Z} \) if \( c \neq 1 \). Since \( c \) is fixed by \( \sigma \), the involution \( \sigma \) induced an anti-holomorphic involution of \( G_c \). This anti-holomorphic involution of \( G_c \) will be denoted by \( \sigma' \). Let \((E_G, \tilde{\alpha})\) be a \((\alpha, \sigma, c)\)-pseudo-real principal \( G\)-bundle on \( X \). Define \( E_{G_c} := E_G/\langle c \rangle \). Note that \( E_{G_c} \) is the principal \( G_c \)-bundle obtained by extending the structure group of \( E_G \) using the quotient homomorphism \( G \to G_c \). The above self-map \( \tilde{\alpha} \) of \( E_G \) descends to a self-map

\[ \tilde{\alpha}' : E_{G_c} \to E_{G_c} \]

Since \( \tilde{\alpha}' \) is an anti-holomorphic map, we have \( \tilde{\alpha}' \circ \tilde{\alpha}' = \text{Id}_{E_{G_c}} \). Therefore, \((E_{G_c}, \tilde{\alpha}')\) is a \((\alpha, \sigma')\)-real principal \( G_c \)-bundle.

The pair \((X, \alpha)\) defines a geometrically irreducible smooth projective curve defined over \( \mathbb{R} \). This curve defined over \( \mathbb{R} \) will be denoted by \( X' \). Assume that \( c \neq 1 \). Let \( G' \)
(respectively, $G'_c$) be the algebraic group, defined over $\mathbb{R}$, given by the pair $(G, \sigma)$ (respectively, $(G_c, \sigma')$). Consider the short exact sequence of sheaves

$$1 \rightarrow (c) = \mathbb{Z}/2\mathbb{Z} \rightarrow G' \rightarrow G'_c \rightarrow 1$$

on $X'$. Let

$$H^1_{\text{ét}}(X', G') \rightarrow H^1_{\text{ét}}(X', G'_c) \overset{\beta}{\rightarrow} H^2_{\text{ét}}(X', \mathbb{Z}/2\mathbb{Z})$$

be the long exact sequence of étale cohomologies corresponding to the above short exact sequence of sheaves on the curve $X'$ defined over $\mathbb{R}$. As noted above, a $(\alpha, \sigma, c)$-pseudo-real principal $G$-bundle on $X$ gives a $(\alpha, \sigma')$-real principal $G_c$-bundle. Note that the isomorphism classes of principal $G'_c$-bundles on $X'$ are parametrized by the elements of the cohomology $H^1_{\text{ét}}(X', G'_c)$. Indeed, this follows immediately from the fact that any principal $G'_c$-bundle on $X'$ can be locally trivialized with respect to the étale topology. Therefore, a $(\alpha, \sigma, c)$-pseudo-real principal $G$-bundle on $X$ gives an element of $H^1_{\text{ét}}(X', G'_c)$.

We will give a necessary and sufficient condition for a given $(\alpha, \sigma')$-real principal $G_c$-bundle on $X$ to come from a $(\alpha, \sigma, c)$-pseudo-real principal $G$-bundle.

Let $(E_{G_c}, \tilde{\alpha}')$ be a $(\alpha, \sigma')$-real principal $G_c$-bundle on $X$. As explained above, $(E_{G_c}, \tilde{\alpha}')$ is equivalently a principal $G'_c$-bundle on $X'$. This principal $G'_c$-bundles on $X'$ will be denoted by $F_{G_c}$. Consider the adjoint action of $G$ on itself. Since $c$ lies in the center of $G$, this action of $G$ factors through the quotient group $G_c$. Let

$$E_{G_c}(G) := E_{G_c} \times^{G_c} G \rightarrow X$$

be the fiber bundle associated to the principal $G_c$-bundle $E_{G_c}$ for this action of $G_c$ on $G$. Since the action of $G_c$ on $G$ preserves the group structure on $G$, each fiber of $E_{G_c}(G)$ is a group isomorphic to $G$. The action of $G_c$ on $G$ descends to an action of $G_c$ on the quotient $G/(c) = G_c$, and this descended action coincides with the adjoint action of $G_c$ on itself. Therefore, the short exact sequence of groups

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow G \rightarrow G_c \rightarrow 1$$

produces a short exact sequence of fiber bundles with group structures

$$1 \rightarrow X \times (\mathbb{Z}/2\mathbb{Z}) \rightarrow E_{G_c}(G) \rightarrow \text{Ad}(E_{G_c}) \rightarrow 1,$$

where $\text{Ad}(E_{G_c}) = E_{G_c} \times^{G_c} G_c$ is the adjoint bundle for $E_{G_c}$.

The involution $\tilde{\alpha}'$ of $E_{G_c}$ and the involution $\sigma$ of $G$ together produce an anti-holomorphic involution of $E_{G_c}(G)$ covering $\alpha$. Similarly, $\tilde{\alpha}'$ and $\sigma'$ together produce an anti-holomorphic involution of $\text{Ad}(E_{G_c})$ covering $\alpha$. Therefore, (3.2) produces a short exact sequence

$$1 \rightarrow X' \times (\mathbb{Z}/2\mathbb{Z}) \rightarrow E_{G_c}(G)' \rightarrow \text{Ad}(E_{G_c})' \rightarrow 1$$

over the curve $X'$ defined over $\mathbb{R}$. We note that $\text{Ad}(E_{G_c})'$ is the adjoint bundle for the principal $G'_c$-bundle $F_{G_c}$ over $X'$ defined by the pair $(E_{G_c}, \tilde{\alpha}')$.

The space of all isomorphism classes of principal $G'_c$-bundles on $X'$ is parametrized by $H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})')$. This identification is constructed as follows. First recall that $\text{Ad}(E_{G_c})'$ is the adjoint bundle for the principal $G'_c$-bundle $F_{G_c}$ over $X'$. Given a principal $G'_c$-bundle on $X'$, by choosing étale local isomorphisms of it with $F_{G_c}$ we
get an element of \( H^1_\rm{ét}(X', \Ad(E_{G_c}')) \). Conversely, given a 1-cocycle on \( X' \) with values in \( \Ad(E_{G_c}') \), by gluing back, using the cocycle, the restrictions of \( F_{G_c} \) to the open subsets for the cocycle, we get a principal \( G'_e \)-bundle on \( X' \). Note that if \( F_{G_c} \) is the trivial principal \( G'_e \)-bundle, then \( H^1_\rm{ét}(X', \Ad(E_{G_c}')) = H^1_\rm{ét}(X', G'_c) \).

The set \( H^1_\rm{ét}(X', \Ad(E_{G_c}')) \) has a distinguished base point \( t_0 \). This point \( t_0 \) corresponds to the isomorphism class of the principal \( G'_e \)-bundle \( F_{G_c} \).

Consider the short exact sequence of étale cohomologies

\[
(3.4) \quad H^1_\rm{ét}(X', E_{G_c}(G)') \xrightarrow{\tau'} H^1_\rm{ét}(X', \Ad(E_{G_c}')) \xrightarrow{\beta'} H^2_\rm{ét}(X', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}
\]

associated to \( (3.3) \). It can be shown that \( (E_{G_c}, \tilde{\alpha}) \) is given by a \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle if and only if the base point \( t_0 \in H^1_\rm{ét}(X', \Ad(E_{G_c}')) \) lies in the image of the map \( \gamma' \) in \( (3.4) \). Indeed, if \( (E_{G_c}, \tilde{\alpha}) \) is a \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle on \( X \) that gives \( (E_{G_c}, \tilde{\alpha}) \), then the adjoint bundle \( \Ad(E_{G_c}) \) equipped with the involution constructed using \( \tilde{\alpha} \) and \( \sigma \) produces an element \( t' \in H^1_\rm{ét}(X', E_{G_c}(G)') \) such that \( \gamma'(t') = t_0 \). Conversely, any \( t' \in H^1_\rm{ét}(X', E_{G_c}(G)') \) produces a \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle. If \( \gamma'(t') = t_0 \), then this \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle gives the pair \((E_{G_c}, \tilde{\alpha})\).

Therefore, we have the following.

**Proposition 3.5.** — A \((\alpha, \sigma')\)-real principal \( G_c \)-bundle \((E_{G_c}, \tilde{\alpha})\) on \( X \) comes from a \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle if and only if \( \beta'(t_0) = 0 \), where \( \beta' \) is the map in \((3.4) \) and \( t_0 \in H^1_\rm{ét}(X', \Ad(E_{G_c}')) \) is the base point.

The following proposition shows the relation between the reality conditions defined by conjugations of \( G \) that are inner equivalent. One has the following.

**Proposition 3.6.** — Let \( \sigma, \sigma' \in \text{Conj}(G) \) such that \( \sigma' = \text{Int}(g_0)\sigma \) for some \( g_0 \in G \), i.e., \( \sigma'(g) = g\sigma(g)g_0^{-1} \). Let \( E \) be a \( G \)-bundle over \( X \). Then \( E \) is \((\alpha, \sigma, c)\)-pseudo-real if and only if it is \((\alpha, \sigma', c')\)-pseudo-real, where \( c \) and \( c' \) are related by \( g_0 \) and \( \sigma \). In fact \( c' = c \), if \( \sigma(g_0) = g_0^{-1} \).

**Proof.** — Let \((E, \tilde{\alpha})\) be a \((\alpha, \sigma, c)\)-pseudo-real principal \( G \)-bundle on \( X \). Define

\[
\tilde{\alpha}' : E \longrightarrow E, \quad e \mapsto \tilde{\alpha}(e)g_0^{-1}.
\]

Since \( \tilde{\alpha} \) is anti-holomorphic and covers \( \alpha \), the map \( \tilde{\alpha}' \) is also anti-holomorphic and covers \( \alpha \). For any \( e \in E \) and \( g \in G \), we have

\[
\tilde{\alpha}'(eg) = \tilde{\alpha}(eg)g_0^{-1} = \tilde{\alpha}(e)\sigma(g)g_0^{-1} = \tilde{\alpha}(e)g_0^{-1}g_0\sigma(g)g_0^{-1} = \tilde{\alpha}'(e)\sigma'(g).
\]

Also,

\[
\tilde{\alpha}'(\tilde{\alpha}'(e)) = \tilde{\alpha}'(\tilde{\alpha}(e)g_0^{-1}) = \tilde{\alpha}(\tilde{\alpha}(e)g_0^{-1})g_0^{-1} = \tilde{\alpha}(\tilde{\alpha}(e))\sigma(g_0^{-1})g_0^{-1} = ecc\sigma(g_0^{-1})g_0^{-1}.
\]

Now, \( \sigma'^2 = \text{Id} \) implies that \( \sigma(g_0^{-1})g_0^{-1} \in \mathbb{Z} \), and we can appeal to Remark 3.3 to claim that by modifying \( \tilde{\alpha}' \) by an element of the center \( c\sigma(g_0^{-1})g_0^{-1} \) is replaced by an element \( c' \in \mathbb{Z}_2^\times \), and hence \( E \) has the structure of a \((\alpha, \alpha', c')\)-pseudo-real principal \( G \)-bundle on \( X \). The last claim in the proposition is clear. \( \square \)
3.3. Pseudo-real $G$-Higgs bundles. — Let $(E, \tilde{\alpha})$ be a $(\alpha, \sigma, c)$-pseudo-real principal $G$-bundle on $X$ as defined above. Let

$$ad(E) := E \times^G g := E(g)$$

be the adjoint vector bundle for $E$. The self-map $\tilde{\alpha}$ of $E$ produces an anti-holomorphic self-map

$$(3.5) \quad \tilde{\alpha}_0 : E(g) \to E(g)$$

such that $q \circ \tilde{\alpha}_0 = \alpha \circ q$, where $q$ is the projection of $E(g)$ to $X$. Since $c \in \mathbb{Z}$, the adjoint action of $c$ on $g$ is trivial. This immediately implies that $\tilde{\alpha}_0$ is an involution.

In other words, $(E(g), \tilde{\alpha}_0)$ is a real vector bundle (see [5]).

The real structure of the canonical line bundle $K$ of $X$ given by $\alpha$ and the above real structure $\tilde{\alpha}_0$ of $E(g)$ combine to define a real structure on the vector bundle $E(g) \otimes K$. For notational convenience, this real structure on $E(g) \otimes K$ will also be denoted by $\tilde{\alpha}$. So

$$\tilde{\alpha} : E(g) \otimes K \to E(g) \otimes K$$

is an anti-holomorphic involution over $\alpha$.

Definition 3.7. — Let $(E, \varphi)$ be a $G$-Higgs bundle. We say that $(E, \varphi)$ is $(\alpha, \sigma, c, +)$-pseudo-real (respectively, $(\alpha, \sigma, c, -)$-pseudo-real) if $E$ is $(\alpha, \sigma, c)$-pseudo-real, and $\varphi$ satisfies

$$\tilde{\alpha}(\varphi) = \varphi \quad \text{(respectively, } \tilde{\alpha}(\varphi) = -\varphi).$$

The concept of $(\alpha, \sigma, c, +)$-pseudo-real Higgs bundle was introduced in [5], where notions of (semi)stability and polystability for these objects were defined. These notions are identical for the $(\alpha, \sigma, c, -)$-pseudo-real case. For the benefit of the reader we recall the basic definitions and facts (see [5] for details).

Let $Ad(E) := E \times^G G$ be the group-scheme over $X$ associated to $E$ for the adjoint action of $G$ on it self. The bundle $Ad(E)$ is equipped with an anti-holomorphic involution

$$(3.6) \quad \tilde{\alpha} : Ad(E) \to Ad(E)$$

(abusing notation again) covering $\alpha$. Note that $\tilde{\alpha}^2 = \text{Id}_{Ad(E)}$ since the adjoint action of $Z^a$ on $G$ is trivial.

A parabolic subgroup scheme of $Ad(E)$ is a Zariski closed analytically locally trivial subgroup scheme $P \subset Ad(E)$ such that $Ad(E)/P$ is compact. For such a parabolic subgroup scheme $P$ let $p \subset \text{ad}(E)$ be the corresponding bundle of Lie algebras.

A $(\alpha, \sigma, c, \pm)$-pseudo-real $G$-Higgs bundle $(E, \varphi, \tilde{\alpha})$ is semistable (respectively, stable) if for every proper parabolic subgroup scheme $P \subset Ad(E)$ such that $\tilde{\alpha}(P) \subset P$, where $\tilde{\alpha}$ is given by (3.6), and $\varphi \in H^0(X, p \otimes K)$,

$$\deg(p) \leq 0 \quad \text{(respectively, } \deg(p) < 0),$$

where $p$ is the vector bundle associated to $P$ defined above.
One has the following (see [5]).

**Proposition 3.8.** — Let \((E, \varphi, \tilde{\alpha})\) be a \((\alpha, \sigma, c, \pm)\)-pseudo-real \(G\)-Higgs bundle.

1. If \((E, \varphi)\) is semistable (respectively stable), in the sense of Section 2.1, then \((E, \varphi, \tilde{\alpha})\) is semistable (respectively stable).
2. If \((E, \varphi, \tilde{\alpha})\) is semistable then \((E, \varphi)\) is semistable.
3. If \((E, \varphi, \tilde{\alpha})\) is stable then \((E, \varphi)\) is polystable (in the sense given in Section 2.1).

To define polystability for a pseudo-real \(G\)-Higgs bundle let \(\mathfrak{p} \subset \text{ad}(E)\) be a parabolic subalgebra bundle such that \(\tilde{\alpha}_0(\mathfrak{p}) = \mathfrak{p}\), where \(\tilde{\alpha}_0\) is the involution defined in (3.5).

Let \(R_u(\mathfrak{p}) \subset \mathfrak{p}\) be the holomorphic subbundle over \(X\) whose fiber over a point \(x \in X\) is the nilpotent radical of the parabolic subalgebra \(\mathfrak{p}_x\). Therefore, the quotient \(\mathfrak{p}/R_u(\mathfrak{p})\) is a bundle of reductive Lie algebras. Note that \(\tilde{\alpha}_0(R_u(\mathfrak{p})) = R_u(\mathfrak{p})\). A Levi subalgebra bundle of \(\mathfrak{p}\) is a holomorphic subbundle \(\ell(\mathfrak{p}) \subset \mathfrak{p}\) such that for each \(x \in X\), the fiber \(\ell(\mathfrak{p})_x\) is a Lie subalgebra of \(\mathfrak{p}_x\), and the composition

\[
\ell(\mathfrak{p}) \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{p}/R_u(\mathfrak{p})
\]

is an isomorphism, where \(\mathfrak{p} \twoheadrightarrow \mathfrak{p}/R_u(\mathfrak{p})\) is the quotient map.

A semistable \((\alpha, \sigma, c, \pm)\)-pseudo-real \(G\)-Higgs bundle \((E, \varphi, \tilde{\alpha})\) is polystable if either is stable, or there is a proper parabolic subalgebra bundle \(\mathfrak{p} \subset \text{ad}(E)\), and a Levi subalgebra bundle \(\ell(\mathfrak{p}) \subset \mathfrak{p}\), such that

\[
\tilde{\alpha}_0(\mathfrak{p}) = \mathfrak{p}, \quad \tilde{\alpha}_0(\ell(\mathfrak{p})) = \ell(\mathfrak{p}), \quad \varphi \in H^0(X, \ell(\mathfrak{p}) \otimes K),
\]

and for every parabolic subalgebra bundle \(\mathfrak{q} \subset \ell(\mathfrak{p})\) with \(\tilde{\alpha}_0(\mathfrak{q}) = \mathfrak{q}\) we have

\[
\text{deg}(\mathfrak{q}) < 0.
\]

We have the following (see [5]).

**Proposition 3.9.** — A \((\alpha, \sigma, c, \pm)\)-pseudo-real \(G\)-Higgs bundle \((E, \varphi, \tilde{\alpha})\) is polystable if and only if \((E, \varphi)\) is polystable.

We can thus define the moduli space \(\mathcal{M}(G, \alpha, \sigma, c, \pm)\) of isomorphism classes of polystable \((\alpha, \sigma, c, \pm)\)-pseudo-real \(G\)-Higgs bundles, and, as a consequence of Proposition 3.9, define maps

\[
\mathcal{M}(G, \alpha, \sigma, c, \pm) \longrightarrow \mathcal{M}(G)
\]

that forget the pseudo-real structure.
4. Involutions of moduli spaces

4.1. Involutions of Higgs bundle moduli spaces. — As before, let $\alpha : X \to X$ and $\sigma : G \to G$ be anti-holomorphic involutions. For a holomorphic principal $G$-bundle $E$ on $X$, let $\sigma(E)$ be the $C^\infty$ principal $G$-bundle on $X$ obtained by extending the structure group of $E$ using the homomorphism $\sigma$. So the total space of $\sigma(E)$ is identified with that of $E$, but the action of $g \in G$ on $e \in E$ coincides with the action of $\sigma(g)$ on $e \in \sigma(E)$. Consequently, the pullback $\alpha^*\sigma(E)$ has a holomorphic structure given by the holomorphic structure of $E$. Let

$$\tilde{\sigma} : E(\mathfrak{g}) \to E(\mathfrak{g})$$

be the conjugate linear isomorphism that sends the equivalence class of any $(e, v) \in E \times \mathfrak{g}$ to the equivalence class of $(e, d\sigma(v))$, where $d\sigma$ is the automorphism of $\mathfrak{g}$ corresponding to $\sigma$. Let $\varphi$ be a Higgs field on $E$. Let $\sigma(\varphi)$ be the $C^\infty$ section of $E(\mathfrak{g}) \otimes K$ defined by $\tilde{\sigma}$ and the $C^\infty$ isomorphism $K \to \overline{K}$ defined by $df \mapsto \overline{df}$, where $f$ is any locally defined holomorphic function on $X$.

We have involutions

$$\iota(\alpha, \sigma) : \mathcal{M}(G) \to \mathcal{M}(G)$$

(4.1)

$$(E, \varphi) \mapsto (\alpha^*\sigma(E), \pm \alpha^*\sigma(\varphi)).$$

Proposition 4.1. — The image of the map

$$\mathcal{M}(G, \alpha, \sigma, c, +) \to \mathcal{M}(G)$$

in (3.7) is contained in the fixed point locus of the involution $\iota(\alpha, \sigma)^+$. Moreover, the fixed point locus of $\iota(\alpha, \sigma)^+$ in the smooth locus $\mathcal{M}(G)^{sm} \subset \mathcal{M}(G)$ is the intersection of $\mathcal{M}(G)^{sm}$ with the union of the images of $\mathcal{M}(G, \alpha, \sigma, c, +)$ as $c$ runs over $\mathbb{Z}_2^n$, where $\mathbb{Z}_2^n$ as before is the subgroup of $\mathbb{Z}^n$ generated by the order two points.

Similarly, the fixed point locus of $\iota(\alpha, \sigma)^-$ in $\mathcal{M}(G)^{sm}$ is the intersection of $\mathcal{M}(G)^{sm}$ with the union of the images of $\mathcal{M}(G, \alpha, \sigma, c, -)$ as $c$ runs over $\mathbb{Z}_2^n$.

Proof. — From the definition of $\iota(\alpha, \sigma)^+$ (respectively, $\iota(\alpha, \sigma)^-$) it follows immediately that $\mathcal{M}(G, \alpha, \sigma, c, +)$ (respectively, $\mathcal{M}(G, \alpha, \sigma, c, -)$) is contained in the fixed point locus of $\iota(\alpha, \sigma)^+$ (respectively, $\iota(\alpha, \sigma)^-$).

A $G$-Higgs bundle $(E, \varphi)$ lies in $\mathcal{M}(G)^{sm}$ if $(E, \varphi)$ is stable and the automorphism group of $(E, \varphi)$ coincides with the center $Z$ of $G$, i.e., if the Higgs bundle is simple as defined in [14, 13, 6] (we recall that such bundles are called regularly stable). Suppose that $(E, \varphi) \in \mathcal{M}(G)^{sm}$ is fixed under the involution $\iota(\alpha, \sigma)^+$ (respectively $\iota(\alpha, \sigma)^-$). This means that there exists an isomorphism

$$f : E \to \alpha^*\sigma(E)$$

such that $\alpha^*\sigma(f) \circ f \in \text{Aut}(E, \varphi)$, but since $\text{Aut}(E, \varphi) = Z$, we have that $\alpha^*\sigma(f) \circ f = c \in Z$. We can interpret $f$ as a map $f' : E \to \sigma(E)$ such that $\sigma(f') \circ f' = c \in Z$. Identifying $\sigma(E)$ with $E$ with multiplication on the right defined by $c \cdot g = c\sigma(g)$, where $g \in G$ and $c \in E$, we are indeed defining a $(\alpha, \sigma, c, +)$ (respectively $(\alpha, \sigma, c, -)$) pseudo-real structure on $(E, \varphi)$, since we can always assume that $c \in Z_2^n$, as explained
in Remark 3.3. In other words, \((E, \varphi)\) lies in the image of \(\mathcal{M}(G, \alpha, \sigma, c, +)\) (respectively \(\mathcal{M}(G, \alpha, \sigma, c, -)\)). □

**Remark 4.2.** — In Definition 3.2 we could have defined a pseudo-real structure replacing \(\tilde{\alpha}\) by an anti-holomorphic map \(\tilde{\alpha}' : E \to \sigma(E)\) of \(G\)-bundles covering \(\alpha\). Although \(\sigma(E)\) is no longer a holomorphic bundle, its total space is a complex manifold because it is identified with the total space of \(E\), and hence the anti-holomorphicity condition makes sense. The condition \(\tilde{\alpha}(eg) = \tilde{\alpha}(e)\sigma(g)\) in Definition 3.2 is now automatic since \(\tilde{\alpha}'\) is a \(G\)-bundle map.

**Proposition 4.3.** — Let \(\sigma\) and \(\sigma'\) be inner equivalent elements in \(\text{Conj}(G)\), i.e., they define the same element in \(\text{Out}_2(G)\). Then

\[
i_\epsilon(\alpha, \sigma)^+ = i_\epsilon(\alpha, \sigma')^+ \quad \text{(respectively, } i_\epsilon(\alpha, \sigma)^- = i_\epsilon(\alpha, \sigma')^-)\]

**Proof.** — If we replace \(\sigma\) by \(\sigma' := g_0\sigma g_0^{-1}\), where \(g_0 \in G\), then the corresponding anti-holomorphic involution of the moduli space is replaced by its composition with the holomorphic automorphism of the moduli space corresponding to the automorphism of \(G\) defined by \(g \mapsto g_0gg_0^{-1}\). But this automorphism of \(G\) produces the identity map of the moduli space. Therefore, the anti-holomorphic involution of the moduli space is unchanged if \(\sigma\) is replaced by \(\sigma'\). □

**Remark 4.4.** — Consider the identification between the \((\alpha, \sigma, c)\)-pseudo-real principal \(G\)-bundles and the \((\alpha, \sigma', c')\)-pseudo-real principal \(G\)-bundles on \(X\) given by Proposition 3.6 when \(\sigma\) and \(\sigma'\) are inner equivalent. Note that a Higgs field on a \((\alpha, \sigma', c')\)-pseudo-real principal \(G\)-bundle produces a Higgs field on the corresponding \((\alpha, \sigma, c)\)-pseudo-real principal \(G\)-bundle, and vice versa. We thus have that by Proposition 3.6 \(\mathcal{M}(G, \alpha, \sigma, c, +)\) is isomorphic to \(\mathcal{M}(G, \alpha, \sigma', c', +)\) (respectively \(\mathcal{M}(G, \alpha, \sigma, c, -)\) is isomorphic to \(\mathcal{M}(G, \alpha, \sigma', c', -)\)) giving the same image under the corresponding maps to \(\mathcal{M}(G)\).

4.2. Correspondence with representations for \(i(\alpha, \sigma)^+\). — We have the orbifold fundamental group of \((X, \alpha)\) that we will denote \(\Gamma(X, \alpha)\) (see [5] for example). This fits into an exact sequence

\[
1 \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(X, x_0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.
\]

Let \(\text{Map}'(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))\) be the space of all maps

\[
\delta : \Gamma(X, x_0) \longrightarrow G \times (\mathbb{Z}/2\mathbb{Z})
\]

such that the following diagram is commutative:

\[
\begin{array}{c}
1 \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(X, x_0) \xrightarrow{\eta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\
\downarrow \delta \quad \downarrow \eta \\
1 \longrightarrow G \longrightarrow G \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.
\end{array}
\]
Take an element $c \in \mathbb{Z}_2^n$ in the subgroup generated by the elements of $\mathbb{Z}^n$ order two. Using it, we will define another group structure on $G \times (\mathbb{Z}/2\mathbb{Z})$. The group operation is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(c)^{e_1}(g_2)^{e_2}, e_1 + e_2).$$

Note that when $c = 1$ we obtain a semidirect product.

Let $\text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z})$ be the space of all maps

$$\delta \in \text{Map}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$$

such that $\delta$ is a homomorphism with respect to this group structure.

Two elements $\delta, \delta' \in \text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ are called equivalent if there is an element $g \in G$ such that $\delta'(z) = g^{-1}\delta(z)g$ for all $z \in \pi_1(X, \alpha)$.

**Theorem 4.5.** — The moduli space $\mathcal{M}(G, \alpha, \sigma, c, +)$ is identified with the space of equivalence classes of reductive elements of $\text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$.

**Proof.** — This follows from Proposition 5.6 of [5].

**Theorem 4.6.** — Consider the involution $\iota(\alpha, \sigma) \dagger$ of $\mathcal{M}(G)$. It is anti-holomorphic with respect to the almost complex structures $J_1$ and $J_2$, and it is holomorphic with respect to $J_3$.

**Proof.** — The almost complex structure $J_1$ is the almost complex structure of the Dolbeault moduli space (the moduli space of Higgs bundles). Therefore, $\iota(\alpha, \sigma) \dagger$ is anti-holomorphic with respect to $J_1$.

The almost complex structure $J_2$ is the almost complex structure of the Betti moduli space (the representation space $\mathcal{R}(G)$). Note that the almost complex structure of the Betti moduli space coincides with that of the de Rham moduli space.

As before, fix a base point $x_0 \in X$. The involution $\alpha$ of $X$ produces an isomorphism

$$\alpha' : \pi_1(X, x_0) \longrightarrow \pi_1(X, \alpha(x_0)).$$

This in turn gives a biholomorphism

$$\alpha'' : \text{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \text{Hom}^+(\pi_1(X, \alpha(x_0)), G)/G.$$ 

As noted before, $\mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G$ is independent of the choice of the base point. So $\alpha''$ is a biholomorphism

$$\alpha'' : \mathcal{R}(G) \longrightarrow \mathcal{R}(G).$$

Since $\alpha$ is an involution, it follows that $\alpha''$ is also an involution.

Let

$$b : \mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathcal{R}(G)$$

be the anti-holomorphic involution defined by $\rho \mapsto \sigma \circ \rho$. In other words, $b$ sends a homomorphism $\rho : \pi_1(X) \rightarrow G$ to the composition

$$\pi_1(X, x_0) \xrightarrow{\rho} G \xrightarrow{\sigma} G.$$ 

Clearly $b$ commutes with the above involution $\alpha''$ in (4.4). Therefore, $b \circ \alpha''$ is also an involution. Note that $b \circ \alpha''$ is anti-holomorphic because $\alpha''$ is holomorphic and $b$ is anti-holomorphic.
The above involution \( \alpha' \circ \alpha'' \) of \( \mathcal{R}(G) \) coincides with the involution \( \iota(\alpha, \sigma)^+ \) of \( \mathcal{M}(G) \) under the correspondence \( \mathcal{M}(G) \cong \mathcal{R}(G) \). Therefore, \( \iota(\alpha, \sigma)^+ \) is anti-holomorphism with respect to \( J_2 \).

We recall that \( J_3 = J_1 J_2 \). Since \( \iota(\alpha, \sigma)^+ \) is anti-holomorphic with respect to both \( J_1 \) and \( J_2 \), from the above identity it follows immediately that \( \iota(\alpha, \sigma)^+ \) is holomorphic with respect to \( J_3 \).

Since \( \mathcal{R}(G) \) is hyper-Kähler, the holomorphic symplectic form \( \Omega_2 \) on it is flat with respect to the Kähler structure \( \omega_3 \) corresponding to \( J_2 \). Similarly, the holomorphic symplectic form \( \Omega_1 \) with respect to \( J_1 \) is flat with respect to the Kähler structure \( \omega_1 \) corresponding to \( J_1 \). In particular, \( \mathcal{R}(G) \) and \( \mathcal{M}(G, J_1, \omega_1, \Omega_1) \) are Calabi-Yau.

**Theorem 4.7.** — The moduli space \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is a special Lagrangian subspace of \( \mathcal{R}(G) \). Similarly, it is special Lagrangian with respect to \( \mathcal{M}(G, J_1, \omega_1, \Omega_1) \). Also, it is complex Lagrangian with respect to \( \mathcal{R}(J_3, \Omega_3) \).

**Proof.** — Since the involution \( \iota(\alpha, \sigma)^+ \) is holomorphic with respect to \( J_3 \), it follows that \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is a holomorphic subspace with respect to \( J_3 \). Recall that \( \Omega_3 = \omega_1 + \sqrt{-1} \omega_2 \). The involution \( \iota(\alpha, \sigma)^+ \) is anti-holomorphic with respect to \( J_1 \) and \( J_2 \) and it is an isometry. Hence \( \iota(\alpha, \sigma)^+ \) takes \( \omega_1 \) and \( \omega_2 \) to \( -\omega_1 \) and \( -\omega_2 \) respectively. Hence \( \iota(\alpha, \sigma)^+ \) takes \( \Omega_3 \) to \( -\Omega_3 \). This immediately implies that \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is Lagrangian with respect to \( \Omega_3 \).

Since \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is the fixed point locus of an isometric anti-holomorphic involution of the Calabi-Yau space \( \mathcal{R}(G) \), it follows that \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is a special Lagrangian subspace of \( \mathcal{R}(G) \). For a similar reason, \( \mathcal{M}(G, \alpha, \sigma, c, +) \) is a special Lagrangian subspace of \( \mathcal{M}(G, J_1, \omega_1) \). \( \square \)

### 4.3. Correspondence with representations for \( \iota(\alpha, \sigma)^- \)

Next we consider the involution \( \iota(\alpha, \sigma)^- \).

Consider the holomorphic involution \( \theta = \sigma \tau \) of \( G \) as defined earlier in Section 3.1. Using \( c \in \mathbb{Z}_2 \), we will define yet another group structure on \( G \times (\mathbb{Z}/2\mathbb{Z}) \). The group operation is given by

\[
(g_1, e_1) \cdot (g_2, e_2) = (g_1(\theta)^c_1 g_2) e^{c_1 e_2}, e_1 + e_2.
\]

Let \( \text{Hom}_-^c(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z})) \) be the space of all maps

\[
\delta \in \text{Map}'(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z}))
\]

such that \( \delta \) is a homomorphism with respect to this new group structure.

Two elements \( \delta', \delta'' \in \text{Hom}_-^c(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z})) \) are called equivalent if there is an element \( g \in G \) such that \( \delta'(z) = g^{-1} \delta(z) g \) for all \( z \in \pi_1(X, \alpha) \).

**Theorem 4.8.** — The moduli space \( \mathcal{M}(G, \alpha, \sigma, c, -) \) is identified with the space of equivalence classes of reductive elements of \( \text{Hom}_-^c(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z})) \).

**Proof.** — This follows from Proposition 5.6 of [5]. \( \square \)
Theorem 4.9. — Consider the involution \( \iota(\alpha, \sigma)^- \) of \( \mathcal{M}(G) \). It is anti-holomorphic with respect to the almost complex structures \( J_1 \) and \( J_3 \), and it is holomorphic with respect to \( J_2 \).

Proof. — The involution \( \iota(\alpha, \sigma)^- \) is clearly anti-holomorphic with respect to \( J_1 \) because \( J_1 \) coincides with the complex structure of the Dolbeault moduli space.

Let \( \tilde{b} : \mathcal{R}(G) = \text{Hom}^+ \left( \pi_1(X, x_0), G \right)/G \to \mathcal{R}(G) \) be the holomorphic involution defined by \( \rho \mapsto \theta \circ \rho \). In other words, \( \tilde{b} \) sends a homomorphism \( \rho : \pi_1(X) \to G \) to the composition

\[
\pi_1(X) \xrightarrow{\rho} G \xrightarrow{\theta} G.
\]

Clearly \( \tilde{b} \) commutes with the above involution \( \alpha'' \) in (4.4). Therefore, \( \tilde{b} \circ \alpha'' \) is also an involution. The composition \( \tilde{b} \circ \alpha'' \) is holomorphic because both \( \alpha'' \) and \( \tilde{b} \) are holomorphic.

The above involution \( \tilde{b} \circ \alpha'' \) of \( \mathcal{R}(G) \) coincides with \( \iota(\alpha, \sigma)^- \), and the complex structure of the Betti moduli space \( \mathcal{R}(G) \) is given by \( J_2 \). Therefore, \( \iota(\alpha, \sigma)^- \) is holomorphic with respect to \( J_2 \).

Since \( J_3 = J_1J_2 \) and \( \iota(\alpha, \sigma)^- \) is anti-holomorphic with respect to \( J_1 \) and holomorphic with respect to \( J_2 \), we conclude that \( \iota(\alpha, \sigma)^- \) is anti-holomorphic with respect to \( J_3 \). □

Consider the complex structure \( J_3 \) and the corresponding holomorphic symplectic form \( \Omega_3 \). Since \( \mathcal{R}(G) \) is hyper-Kähler, \( \Omega_3 \) is flat with respect to the Kähler structure for \( J_3 \). Now we have following analog of Theorem 4.7.

Theorem 4.10. — The moduli space \( \mathcal{M}(G, \alpha, \sigma, c, -) \) is a special Lagrangian subspace of \( (\mathcal{M}(G), J_1, \omega_1, \Omega_1) \). Similarly, it is special Lagrangian with respect to \( (\mathcal{M}(G), J_3, \omega_3, \Omega_3) \). Also, it is a complex Lagrangian subspace with respect to \( (\mathcal{R}(G), J_2, \Omega_2) \).

Corollary 4.11. — The fixed point locus of the involution \( \iota(\alpha, \sigma)^- \) is a complex subspace of \( \mathcal{M}(G) \) with the complex structure induced by \( J_2 \), i.e., the natural complex structure of the moduli space of representations \( \mathcal{R}(G) \).

Remark 4.12. — Corollary 4.11 is obtained by Baraglia–Schaposnik [3] in the case when \( \sigma \) is the compact conjugation \( \tau \).

References

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