Arnaud Beauville

Some surfaces with maximal Picard number


<http://jep.cedram.org/item?id=JEP_2014__1___101_0>


Certains droits réservés.

[CC BY-ND] Cet article est mis à disposition selon les termes de la licence
Creative Commons attribution – pas de modification 3.0 France.
http://creativecommons.org/licenses/by-nd/3.0/fr/

L’accès aux articles de la revue « Journal de l’École polytechnique — Mathématiques »
(http://jep.cedram.org/), implique l’accord avec les conditions générales d’utilisation
(http://jep.cedram.org/legal/).
SOME SURFACES WITH MAXIMAL PICARD NUMBER

by Arnaud Beauville

Abstract. — For a smooth complex projective variety, the rank $\rho$ of the Néron-Severi group is bounded by the Hodge number $h^{1,1}$. Varieties with $\rho = h^{1,1}$ have interesting properties, but are rather sparse, particularly in dimension 2. We discuss in this note a number of examples, in particular those constructed from curves with special Jacobians.

Résumé (Quelques surfaces dont le nombre de Picard est maximal). — Le rang $\rho$ du groupe de Néron-Severi d’une variété projective lisse complexe est borné par le nombre de Hodge $h^{1,1}$. Les variétés satisfaisant à $\rho = h^{1,1}$ ont des propriétés intéressantes, mais sont assez rares, particulièrement en dimension 2. Dans cette note nous analysons un certain nombre d’exemples, notamment ceux construits à partir de courbes à jacobienne spéciale.

Contents

1. Introduction .................................................................. 101
2. Generalities ................................................................... 102
3. Abelian varieties .............................................................. 103
4. Products of curves ............................................................ 104
5. Quotients of self-products of curves ........................................... 109
6. Other examples ............................................................... 111
7. The complex torus associated to a $\rho$-maximal variety ................. 112
8. Higher codimension cycles .................................................... 114
References ....................................................................... 115

1. Introduction

The Picard number of a smooth projective variety $X$ is the rank $\rho$ of the Néron-Severi group — that is, the group of classes of divisors in $H^2(X, \mathbb{Z})$. It is bounded by the Hodge number $h^{1,1} := \dim H^1(X, \Omega^1_X)$. We are interested here in varieties with maximal Picard number $\rho = h^{1,1}$. As we will see in §2, there are many examples of such varieties in dimension $\geq 3$, so we will focus on the case of surfaces.

Keywords. — Algebraic surfaces, Picard group, Picard number, curve correspondences, Jacobians.

ISSN: 2270-518X http://jep.cedram.org/
Apart from the well understood case of K3 and abelian surfaces, the quantity of known examples is remarkably small. In [Per82] Persson showed that some families of double coverings of rational surfaces contain surfaces with maximal Picard number (see Section 6.4 below); some scattered examples have appeared since then. We will review them in this note and examine in particular when the product of two curves has maximal Picard number – this provides some examples, unfortunately also quite sparse.

2. Generalities

Let $X$ be a smooth projective variety over $\mathbb{C}$. The Néron-Severi group $\text{NS}(X)$ is the subgroup of algebraic classes in $H^2(X, \mathbb{Z})$; its rank $\rho$ is the Picard number of $X$. The natural map $\text{NS}(X) \otimes \mathbb{C} \to H^2(X, \mathbb{C})$ is injective and its image is contained in $H^{1,1}$, hence $\rho \leq h^{1,1}$.

Proposition 1. — The following conditions are equivalent:

(i) $\rho = h^{1,1}$;

(ii) The map $\text{NS}(X) \otimes \mathbb{C} \to H^{1,1}$ is bijective;

(iii) The subspace $H^{1,1}$ of $H^2(X, \mathbb{C})$ is defined over $\mathbb{Q}$.

(iv) The subspace $H^{2,0} \oplus H^{0,2}$ of $H^2(X, \mathbb{C})$ is defined over $\mathbb{Q}$.

Proof. — The equivalence of (iii) and (iv) follows from the fact that $H^{2,0} \oplus H^{0,2}$ is the orthogonal of $H^{1,1}$ for the scalar product on $H^2(X, \mathbb{C})$ associated to an ample class. The rest is clear.

When $X$ satisfies these equivalent properties we will say for short that $X$ is $\rho$-maximal (one finds the terms singular, exceptional or extremal in the literature).

Remarks

(1) A variety with $H^{2,0} = 0$ is $\rho$-maximal. We will implicitly exclude this trivial case in the discussion below.

(2) Let $X$, $Y$ be two $\rho$-maximal varieties, with $H^1(Y, \mathbb{C}) = 0$. Then $X \times Y$ is $\rho$-maximal. For instance $X \times \mathbb{P}^n$ is $\rho$-maximal, and $Y \times C$ is $\rho$-maximal for any curve $C$.

(3) Let $Y$ be a submanifold of $X$; if $X$ is $\rho$-maximal and the restriction map $H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C})$ is bijective, $Y$ is $\rho$-maximal. By the Lefschetz theorem, the latter condition is realized if $Y$ is a complete intersection of smooth ample divisors in $X$, of dimension $\geq 3$. Together with Remark 2, this gives many examples of $\rho$-maximal varieties of dimension $\geq 3$; thus we will focus on finding $\rho$-maximal surfaces.

Proposition 2. — Let $\pi : X \longrightarrow Y$ be a rational map of smooth projective varieties.

(a) If $\pi^* : H^{2,0}(Y) \to H^{2,0}(X)$ is injective (in particular if $\pi$ is dominant), and $X$ is $\rho$-maximal, so is $Y$.

(b) If $\pi^* : H^{2,0}(Y) \to H^{2,0}(X)$ is surjective and $Y$ is $\rho$-maximal, so is $X$. 

J.É.P. - M., 2014, tome 1
Some surfaces with maximal Picard number

Note that since $\pi$ is defined on an open subset $U \subset X$ with $\text{codim}(X \setminus U) \geq 2$, the pull back map $\pi^* : H^2(Y, \mathbb{C}) \to H^2(U, \mathbb{C}) \cong H^2(X, \mathbb{C})$ is well defined.

**Proof.** — Hironaka’s theorem provides a diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{b} & Y \\
\pi & \downarrow & \\
X & \xrightarrow{\pi} & Y
\end{array}
\]

where $\hat{\pi}$ is a morphism, and $b$ is a composition of blowing-ups with smooth centers. Then $b^* : H^{2,0}(X) \to H^{2,0}(\hat{X})$ is bijective, and $\hat{X}$ is $\rho$-maximal if and only if $X$ is $\rho$-maximal; so replacing $\pi$ by $\hat{\pi}$ we may assume that $\pi$ is a morphism.

(a) Let $V := (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{Q})$. We have

\[
V \otimes \mathbb{Q} \cong (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{C}) = (\pi^*)^{-1}(H^{1,1}(X)) = H^{1,1}(Y)
\]

the last equality holds because $\pi^*$ is injective on $H^{2,0}(Y)$ and $H^{0,2}(Y)$, hence $Y$ is $\rho$-maximal.

(b) Let $W$ be the $\mathbb{Q}$-vector subspace of $H^2(Y, \mathbb{Q})$ such that

\[
W \otimes \mathbb{Q} \cong H^{2,0}(Y) \oplus H^{0,2}(Y).
\]

Then $\pi^*W$ is a $\mathbb{Q}$-vector subspace of $H^2(X, \mathbb{Q})$, and

\[
(\pi^*W) \otimes \mathbb{C} = \pi^*(W \otimes \mathbb{C}) = \pi^*H^{2,0}(Y) \oplus \pi^*H^{0,2}(Y) = H^{2,0}(X) \oplus H^{0,2}(X),
\]

so $X$ is $\rho$-maximal. □

3. Abelian varieties

There is a nice characterization of $\rho$-maximal abelian varieties ([Kat75], [Lan75]):

**Proposition 3.** — Let $A$ be an abelian variety of dimension $g$. We have

\[
\text{rk}_\mathbb{Z} \text{End}(A) \leq 2g^2.
\]

The following conditions are equivalent:

(i) $A$ is $\rho$-maximal;

(ii) $\text{rk}_\mathbb{Z} \text{End}(A) = 2g^2$;

(iii) $A$ is isogenous to $E^g$, where $E$ is an elliptic curve with complex multiplication.

(iv) $A$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

(The equivalence of (i), (ii) and (iii) follows easily from Lemma 1 below; the only delicate point is (iii) $\Rightarrow$ (iv), which we will not use.)

Coming back to the surface case, suppose that our abelian variety $A$ contains a surface $S$ such that the restriction map $H^{2,0}(A) \to H^{2,0}(S)$ is surjective. Then $S$ is $\rho$-maximal if $A$ is $\rho$-maximal (Proposition 2(b)). Unfortunately this situation seems to be rather rare. We will discuss below (Proposition 6) the case of $\text{Sym}^2 C$ for a curve $C$. Another interesting example is the Fano surface $F_X$ parametrizing the lines
contained in a smooth cubic threefold $X$, embedded in the intermediate Jacobian $JX$ [CG72]. There are some cases in which $JX$ is known to be $\rho$-maximal:

**Proposition 4**

(a) For $\lambda \in \mathbb{C}$, $\lambda^2 \neq 1$, let $X_\lambda$ (resp. $E_\lambda$) be the cubic in $\mathbb{P}^4$ (resp. $\mathbb{P}^2$) defined by $X_\lambda : X^3 + Y^3 + Z^3 - 3\lambda XYZ + T^3 + U^3 = 0$, $E_\lambda : X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0$. If $E_\lambda$ is isogenous to $E_0$, $JX_\lambda$ and $F_{X_\lambda}$ are $\rho$-maximal. The set of $\lambda \in \mathbb{C}$ for which this happens is countably infinite.

(b) Let $X \subset \mathbb{P}^4$ be the Klein cubic threefold $\sum_{i\in\mathbb{Z}/3} X_i^2 X_{i+1} = 0$. Then $JX$ and $F_X$ are $\rho$-maximal.

**Proof.** — Part (a) is due to Rouleau [Rou11], who proves that $JX_\lambda$ (for any $\lambda$) is isogenous to $E_0^3 \times E_0^3$. Since the family $(E_\lambda)_{\lambda \in \mathbb{C}}$ is not constant, there is a countably infinite set of $\lambda \in \mathbb{C}$ for which $E_\lambda$ is isogenous to $E_0$, hence $JX_\lambda$ and therefore $F_{X_\lambda}$ are $\rho$-maximal.

Part (b) follows from a result of Adler [Adl81], who proves that $JX$ is isogenous (actually isomorphic) to $E^5$, where $E$ is the elliptic curve whose endomorphism ring is the ring of integers of $\mathbb{Q}(\sqrt{-1})$ (see also [Rou09] for a precise description of the group NS$(X)$).

\[ \square \]

4. Products of curves

**Proposition 5.** — Let $C, C'$ be two smooth projective curves, of genus $g$ and $g'$ respectively. The following conditions are equivalent:

(i) The surface $C \times C'$ is $\rho$-maximal;

(ii) There exists an elliptic curve $E$ with complex multiplication such that $JC$ is isogenous to $E^g$ and $JC'$ to $E^{g'}$.

**Proof.** — Let $p, p'$ be the projections from $C \times C'$ to $C$ and $C'$. We have

$$H^{1,1}(C \times C') = p^* H^2(C, \mathbb{C}) \oplus p'^* H^2(C', \mathbb{C}) \oplus (p^* H^{1,0}(C) \otimes p'^* H^{0,1}(C')) \oplus (p^* H^{0,1}(C) \oplus p'^* H^{1,0}(C')),$$

hence $h^{1,1}(C \times C') = 2gg' + 2$. On the other hand we have

$$\text{NS}(C \times C') = p^* \text{NS}(C) \oplus p'^* \text{NS}(C') \oplus \text{Hom}(JC, JC').$$

([LB92], Th. 11.5.1), hence $C \times C'$ is $\rho$-maximal if and only if $\text{rk Hom}(JC, JC') = 2gg'$. Thus the Proposition follows from the following (well-known) lemma:

**Lemma 1.** — Let $A$ and $B$ be two abelian varieties, of dimension $a$ and $b$ respectively. The $\mathbb{Z}$-module $\text{Hom}(A, B)$ has rank $\leq 2ab$; equality holds if and only if there exists an elliptic curve $E$ with complex multiplication such that $A$ is isogenous to $E^a$ and $B$ to $E^b$.
Proof: There exist simple abelian varieties $A_1, \ldots, A_s$, with distinct isogeny classes, and nonnegative integers $p_1, \ldots, p_s, q_1, \ldots, q_s$ such that $A$ is isogenous to $A_1^{p_1} \times \cdots \times A_s^{p_s}$ and $B$ to $A_1^{q_1} \times \cdots \times A_s^{q_s}$. Then

$$\text{Hom}(A, B) \otimes \mathbb{Q} \cong M_{p_1 q_1}(K_1) \times \cdots \times M_{p_s q_s}(K_s),$$

where $K_i$ is the (possibly skew) field $\text{End}(A_i) \otimes \mathbb{Q}$. Put $a_i := \dim A_i$. Since $K_i$ acts on $H^1(A_i, \mathbb{Q})$ we have $\dim_{\mathbb{Q}} K_i \leq b_1(A_i) = 2a_i$, hence

$$\text{rk} \text{Hom}(A, B) \leq \sum_i 2p_i q_i a_i \leq 2 \left( \sum p_i a_i \right) \left( \sum q_i a_i \right) = 2ab.$$

The last inequality is strict unless $s = a_1 = 1$, in which case the first one is strict unless $\dim_{\mathbb{Q}} K_1 = 2$. The lemma, and therefore the Proposition, follow. □

The most interesting case occurs when $C = C'$. Then:

**Proposition 6.** Let $C$ be a smooth projective curve. The following conditions are equivalent:

(i) The Jacobian $JC$ is $\rho$-maximal;
(ii) The surface $C \times C$ is $\rho$-maximal;
(iii) The symmetric square $\text{Sym}^2 C$ is $\rho$-maximal.

Proof: The equivalence of (i) and (ii) follows from Proposition 5. The Abel-Jacobi map $\text{Sym}^2 C \to JC$ induces an isomorphism

$$H^{2,0}(JC) \cong \wedge^2 H^0(C, K_C) \cong H^{2,0}(\text{Sym}^2 C),$$

thus (i) and (iii) are equivalent by Proposition 2. □

When the equivalent conditions of Proposition 6 hold, we will say that $C$ has maximal correspondences (the group $\text{End}(JC)$ is often called the group of divisorial correspondences of $C$).

By Proposition 3 the Jacobian $JC$ is then isomorphic to a product of isogenous elliptic curves with complex multiplication. Though we know very few examples of such curves, we will give below some examples with $g = 4$ or 10.

For $g = 2$ or 3, there is a countably infinite set of curves with maximal correspondences ([HN65], [Hof91]). The point is that any indecomposable principally polarized abelian variety of dimension 2 or 3 is a Jacobian; thus it suffices to construct an indecomposable principal polarization on $E^g$, where $E$ is an elliptic curve with complex multiplication, and this is easily translated into a problem about hermitian forms of rank $g$ on certain rings of quadratic integers.

This approach works only for $g = 2$ or 3; moreover it does not give an explicit description of the curves. Another method is by using automorphism groups, with the help of the following easy lemma:
Lemma 2. — Let $G$ be a finite group of automorphisms of $C$, and let $H^0(C, K_C) = \oplus_{i \in I} V_i$ be a decomposition of the $G$-module $H^0(C, K_C)$ into irreducible representations. Assume that there exists an elliptic curve $E$ and for each $i \in I$, a nontrivial map $\pi_i : C \to E$ such that $\pi_i^* H^0(E, K_E) \subset V_i$. Then $JC$ is isogenous to $E^g$.

In particular if $H^0(C, K_C)$ is an irreducible $G$-module and $C$ admits a map onto an elliptic curve $E$, then $JC$ is isogenous to $E^g$.

Proof. — Let $\eta$ be a generator of $H^0(E, K_E)$. Let $i \in I$; the forms $g^* \pi_i^* \eta$ for $g \in G$ generate $V_i$, hence there exists a subset $A_i$ of $G$ such that the forms $g^* \pi_i^* \eta$ for $g \in A_i$ form a basis of $V_i$.

Put $\Pi_i = (g \circ \pi_i)_{g \in A_i} : C \to E^{A_i}$, and $\Pi = (\Pi_i)_{i \in I} : C \to E^g$. By construction $\Pi^* : H^0(E^g, \Omega^{\otimes 2}) \to H^0(C, K_C)$ is an isomorphism. Therefore the map $JC \to E^g$ deduced from $\Pi$ is an isogeny. \hfill $\square$

In the examples which follow, and in the rest of the paper, we put $\omega := e^{2\pi i/3}$.

Example 1. — We consider the family $(C_t)$ of genus 2 curves given by $y^2 = x^6 + tx^3 + 1$, for $t \in \mathbb{C} \setminus \{\pm 2\}$. It admits the automorphisms

$$\tau : (x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^3}\right) \quad \text{and} \quad \psi : (x, y) \mapsto (\omega x, y).$$

The forms $dx/y$ and $xdx/y$ are eigenvectors for $\psi$ and are exchanged (up to sign) by $\tau$; it follows that the action of the group generated by $\psi$ and $\tau$ on $H^0(C_t, K_{C_t})$ is irreducible.

Let $E_t$ be the elliptic curve defined by $v^2 = (u+2)(u^3-3u+t)$; the curve $C_t$ maps onto $E_t$ by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{y(x+1)}{x^2}\right).$$

By Lemma 2 $JC_t$ is isogenous to $E_t^g$. Since the $j$-invariant of $E_t$ is a non-constant function of $t$, there is a countably infinite set of $t \in \mathbb{C}$ for which $E_t$ has complex multiplication, hence $C_t$ has maximal correspondences.

Example 2. — Let $C$ be the genus 2 curve $y^2 = x(x^4-1)$; its automorphism group is a central extension of $\mathfrak{S}_4$ by the hyperelliptic involution $\sigma$ ([LB92], 11.7); its action on $H^0(C, K_C)$ is irreducible.

Let $E$ be the elliptic curve $E : v^2 = u(u+1)(u-2\alpha)$, with $\alpha = 1 - \sqrt{2}$. The curve $C$ maps to $E$ by

$$(x, y) \mapsto \left(\frac{x^2 + 1}{x-1}, \frac{y(x-\alpha)}{(x-1)^2}\right).$$

The $j$-invariant of $E$ is 8000, so $E$ is the elliptic curve $\mathbb{C}/\mathbb{Z}[\sqrt{2}]$ ([Sil94], Prop. 2.3.1).

Example 3 (The $\mathfrak{S}_4$-invariant quartic curves). — Consider the standard representation of $\mathfrak{S}_4$ on $\mathbb{C}^3$. It is convenient to view $\mathfrak{S}_4$ as the semi-direct product $(\mathbb{Z}/2)^2 \rtimes \mathfrak{S}_3$, 

J.E.P. — M., 2014, tome 1
with $\mathfrak{S}_3$ (resp. $(\mathbb{Z}/2)^2$) acting on $\mathbb{C}^3$ by permutation (resp. change of sign) of the basis vectors. The quartic forms invariant under this representation form the pencil

$$(C_t)_{t \in \mathbb{P}^1}: x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$  

According to [DK93], this pencil was known to Ciani. It contains the Fermat quartic $(t = 0)$ and the Klein quartic $(t = \frac{1}{2}(1 \pm i\sqrt{7}))$.

Let us take $t \notin \{2, -1, -2, \infty\}$; then $C_t$ is smooth. The action of $\mathfrak{S}_4$ on $H^0(C_t, K)$, given by the standard representation, is irreducible. Moreover the involution $x \mapsto -x$ has 4 fixed points, hence the quotient curve $E_t$ has genus 1. It is given by the degree 4 equation

$$u^2 + tu(y^2 + z^2) + y^4 + z^4 + ty^2z^2 = 0$$

in the weighted projective space $\mathbb{P}(2, 1, 1)$. Thus $E_t$ is a double covering of $\mathbb{P}^1$ branched along the zeroes of the polynomial $(t + 2)(y^4 + z^4) + 2ty^2z^2$. The cross-ratio of these zeroes is $-(t + 1)$, so $E_t$ is the elliptic curve $y^2 = x(x - 1)(x + t + 1)$. By Lemma 2 $JC_t$ is isogenous to $E_t^3$. For a countably infinite set of $t$ the curve $E_t$ has complex multiplication, thus $C_t$ has maximal correspondences. For $t = 0$ we recover the well known fact that the Jacobian of the Fermat quartic curve is isogenous to $(\mathbb{C}/\mathbb{Z}[i])^3$.

**Example 4.** Consider the genus 3 hyperelliptic curve $H : y^2 = x(x^6 + 1)$. The space $H^0(H, K_H)$ is spanned by $dx/y$, $xdx/y$, $x^2dx/y$. This is a basis of eigenvectors for the automorphism $\tau : (x, y) \mapsto (\omega x, \omega^2 y)$. On the other hand the involution $\sigma : (x, y) \mapsto (1/x, -y/x^4)$ exchanges $dx/y$ and $x^2dx/y$, hence the summands of the decomposition

$$H^0(H, K_H) = \langle \frac{dx}{y}, x^2\frac{dx}{y} \rangle \oplus \langle \frac{x}{y} \rangle$$

are irreducible under the group $\mathfrak{S}_3$ generated by $\sigma$ and $\tau$.

Let $E_t$ be the elliptic curve $u^2 = u^4 + u$, with endomorphism ring $\mathbb{Z}[i]$. Consider the maps $f$ and $g$ from $H$ to $E_t$ given by

$$f(x, y) = (x^2, xy) \quad g(x, y) = \left(\lambda^2\left(x + \frac{1}{x}\right), \frac{\lambda^3y}{x^2}\right) \quad \text{with} \quad \lambda^{-4} = -3.$$ 

We have

$$f^* \frac{du}{v} = \frac{2xdx}{y} \quad \text{and} \quad g^* \frac{du}{v} = \lambda^{-1}(x^2 - 1)\frac{dx}{y}.$$ 

Thus we can apply Lemma 2, and we find that $JH$ is isogenous to $E_t^3$.

Thus $JH$ is isogenous to the Jacobian of the Fermat quartic $F_4$ (Example 3). In particular we see that the surface $H \times F_4$ is $\rho$-maximal.

We now arrive to our main example in higher genus. Recall that we put $\omega = e^{2\pi i/3}$.

**Proposition 7.** The Fermat sextic curve $C_6 : X^6 + Y^6 + Z^6 = 0$ has maximal correspondences. Its Jacobian $JC_6$ is isogenous to $E_\omega^{10}$, where $E_\omega$ is the elliptic curve $\mathbb{C}/\mathbb{Z}[\omega]$. 

\[J.É.P – M., 2014, tome 1\]
The first part can be deduced from the general recipe given by Shioda to compute the Picard number of $C_d \times C_d$ for any $d$ [Shi81]. Let us give an elementary proof. Let $G := T \rtimes \mathfrak{S}_3$, where $\mathfrak{S}_3$ acts on $\mathbb{C}^3$ by permutation of the coordinates and $T$ is the group of diagonal matrices $t$ with $t^6 = 1$.

Let
\[
\Omega = \frac{X dY - Y dX}{Z^5} = \frac{Y dZ - Z dY}{X^5} = \frac{Z dX - X dZ}{Y^5} \in H^0(C, K_C(-3)).
\]
A basis of eigenvectors for the action of $T$ on $H^0(C_6, K)$ is given by the forms $X^a Y^b Z^c \Omega$, with $a + b + c = 3$; using the action of $\mathfrak{S}_3$ we get a decomposition into irreducible components:
\[
H^0(C_6, K) = V_{3,0,0} \oplus V_{2,1,0} \oplus V_{1,1,1},
\]
where $V_{\alpha, \beta, \gamma}$ is spanned by the forms $X^a Y^b Z^c \Omega$ with $\{a, b, c\} = \{\alpha, \beta, \gamma\}$.

Let us use affine coordinates $x = X/Z$, $y = Y/Z$ on $C_6$. We consider the following maps from $C_6$ onto $E_\alpha$: $v^2 = u^3 - 1$:
\[
f(x, y) = (-x^2, y^3), \quad g(x, y) = \left(2^{-2/3} x^{-2} y^4, \frac{1}{2} (x^3 - x^{-3})\right);
\]
and, using for $E_\alpha$ the equation $e^3 + p^3 + 1 = 0$, $h(x, y) = (x^2, y^2)$.

We have
\[
f^* \frac{du}{v} = -\frac{2xdx}{y^5} = -2XYZ \Omega \in V_{2,1,0},
\]
\[
g^* \frac{du}{v} = -2^{1/3} Y^3 \Omega \in V_{3,0,0},
\]
\[
h^* \frac{d\xi}{\eta^3} = 2XYZ \Omega \in V_{1,1,1},
\]
so the Proposition follows from Lemma 2. \qed

By Proposition 2 every quotient of $C_6$ has again maximal correspondences. There are four such quotient which have genus 4:

- The quotient by an involution $\alpha \in T$, which we may take to be $\alpha : (X, Y, Z) \mapsto (X, Y, -Z)$. The canonical model of $C_6/\alpha$ is the image of $C_6$ by the map
\[
(X, Y, Z) \mapsto (X^2, XY, Y^2, Z^2);
\]
its equations in $\mathbb{P}^3$ are $xz - y^2 = x^3 + z^3 + t^3 = 0$. Projecting onto the conic $xz - y^2 = 0$ realizes $C_6/\alpha$ as the cyclic triple covering $v^3 = u^6 + 1$ of $\mathbb{P}^1$.

- The quotient by an involution $\beta \in \mathfrak{S}_3$, say $\beta : (X, Y, Z) \mapsto (Y, X, Z)$. The canonical model of $C_6/\beta$ is the image of $C_6$ by the map
\[
(X, Y, Z) \mapsto ((X + Y)^2, Z(X + Y), Y^2, XZ);
\]
its equations are $xz - y^2 = x(x - 3t)^2 + z^3 - 2t^3 = 0$.

Since the quadric containing their canonical model is singular, the two genus 4 curves $C_6/\alpha$ and $C_6/\beta$ have a unique $g^1_3$. The associated triple covering $C_6/\alpha \to \mathbb{P}^1$ is cyclic, while the corresponding covering $C_6/\beta \to \mathbb{P}^1$ is not. Therefore the two curves are not isomorphic.
• The quotient by an element of order 3 of $T$ acting freely, say $\gamma : (X,Y,Z) \mapsto (X,\omega Y,\omega^2 Z)$. The canonical model of $C_6/\gamma$ is the image of $C_6$ by the map
$$(X,Y,Z) \mapsto (X^3,Y^3,Z^3,XYZ);$$
its equations are $x^2+y^2+z^2 = t^3-xyz = 0$. Projecting onto the conic $x^2+y^2+z^2 = 0$ realizes $C_6/\gamma$ as the cyclic triple covering $v^3 = u(u^4-1)$ of $\mathbb{P}^1$; thus $C_6/\gamma$ is not isomorphic to $C_6/\alpha$ or $C_6/\beta$.

• The quotient by an element of order 3 of $S_3$ acting freely, say $\delta : (X,Y,Z) \mapsto (Y,Z,X)$. The canonical model of $C_6/\delta$ is the image of $C_6$ by the map
$$(X,Y,Z) \mapsto (X^3+Y^3+Z^3,XYZ,X^2Y+Y^2Z+Z^2X,XY^2+YZ^2+ZX^2).$$
It is contained in the smooth quadric $(x+y)^2+5y^2-2zt = 0$, so $C_6/\delta$ is not isomorphic to any of the 3 previous curves.

Thus we have found four non-isomorphic curves of genus 4 with Jacobian isogenous to $E_4\omega$. The product of any two of these curves is a $\rho$-maximal surface.

Corollary 1. — The Fermat sextic surface $S_6$: $X^6+Y^6+Z^6+T^6 = 0$ is $\rho$-maximal.

Proof. — This follows from Propositions 7.2 and Shioda’s trick: there exists a rational dominant map $\pi : C_6 \times C_6 \dashrightarrow S_6$, given by
$$\pi((X,Y,Z),(X',Y',Z')) = (XZ',YZ',iX'Z,iY'Z).$$

Remark 4. — Since the Fermat plane quartic has maximal correspondences (Example 2), the same argument gives the classical fact that the Fermat quartic surface is $\rho$-maximal. It follows from the explicit formula for $\rho(S_d)$ given in [Aok83] that $S_d$ is $\rho$-maximal (for $d \geq 4$) only for $d = 4$ and 6.

Again every quotient of the Fermat sextic is $\rho$-maximal. For instance, the quotient of $S_6$ by the automorphism $(X,Y,Z,T) \mapsto (X,Y,Z,\omega T)$ is the double covering of $\mathbb{P}^2$ branched along $C_6$; it is a $\rho$-maximal K3 surface. The quotient of $S_6$ by the involution $(X,Y,Z,T) \mapsto (X,Y,-Z,-T)$ is given in $\mathbb{P}^5$ by the equations
$$y^2-xz = v^2-wu = x^3+z^3+u^3+w^3 = 0;$$
it is a complete intersection of degrees $(2,2,3)$, with 12 ordinary nodes. Other quotients have $p_g$ equal to 2, 3, 4 or 6.

5. Quotients of self-products of curves

The method of the previous section may sometimes allow to prove that certain quotients of a product $C \times C$ have maximal Picard number. Since we have very few examples we will refrain from giving a general statement and contend ourselves with one significant example.

Let $C$ be the curve in $\mathbb{P}^4$ defined by
$$u^2 = xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$
It is isomorphic to the modular curve $X(8)$ [FSM13]. Let $\Gamma \subset \text{PGL}(5, \mathbb{C})$ be the subgroup of diagonal elements changing an even number of signs of $u, v, w$; $\Gamma$ is isomorphic to $(\mathbb{Z}/2)^2$ and acts freely on $C$.

**Proposition 8**

(a) $JC$ is isogenous to $E_1^3 \times E_2^{2 \sqrt{-2}}$, where $E_\alpha = \mathbb{C}/\mathbb{Z}[\alpha]$ for $\alpha = i$ or $\sqrt{-2}$.

(b) The surface $(C \times C)/\Gamma$ is $\rho$-maximal.

**Proof**

(a) The form $\Omega := (xy - ydx)/uvw$ generates $H^0(C, K_{C}(-1))$, and is $\Gamma$-invariant; thus multiplication by $\Omega$ induces a $\Gamma$-equivariant isomorphism

$$H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(1)) \xrightarrow{\sim} H^0(C, K_{C}).$$

Let $V$ and $L$ be the subspaces of $H^0(C, K_{C})$ corresponding to $\langle u, v, w \rangle$ and $\langle x, y \rangle$. The projection $\langle u, v, w, x, y \rangle \mapsto \langle u, v, w \rangle$ maps $C$ onto the quartic curve $F: 4u^4 + v^4 - w^4 = 0$; the induced map $f : C \to F$ identifies $F$ with the quotient of $C$ by the involution $(u, v, w, x, y) \mapsto (u, v, w, -x, -y)$, and we have $f^*H^0(F, K_{F}) = V$.

The quotient curve $H := C/\Gamma$ is the genus 2 curve $z^2 = t(t^4 - 1)$ [Bea13]. The pull-back of $H^0(H, K_{H})$ is the subspace invariant under $\Gamma$, that is $L$. Thus $JC$ is isogenous to $JF \times JH$. From examples 1 and 2 of §4 we conclude that $JC$ is isogenous to $E_1^3 \times E_2^{2 \sqrt{-2}}$.

(b) We have $\Gamma$-equivariant isomorphisms

$$H^{1,1}(C \times C) = H^2(C, \mathbb{C}) \oplus H^2(C, \mathbb{C}) \oplus (H^{1,0} \boxtimes H^{0,1}) \oplus (H^{0,1} \boxtimes H^{1,0})$$

$$= \mathbb{C}^2 \oplus \text{End}(H^0(C, K_{C}))^\oplus 2$$

(where $\Gamma$ acts trivially on $\mathbb{C}^2$), hence

$$H^{1,1}((C \times C)/\Gamma) = \mathbb{C}^2 \oplus \text{End}_{\Gamma}(H^0(C, K_{C}))^\oplus 2.$$

As a $\Gamma$-module we have $H^0(C, K_{C}) = L \oplus V$, where $\Gamma$ acts trivially on $L$ and $V$ is the sum of the 3 nontrivial one-dimensional representations of $\Gamma$. Thus

$$\text{End}_{\Gamma}(H^0(C, K_{C})) = M_2(\mathbb{C}) \times \mathbb{C}^3.$$  

Similarly we have $\text{NS}((C \times C)/\Gamma) \otimes \mathbb{Q} = \mathbb{Q}^2 \oplus (\text{End}_{\Gamma}(JC) \otimes \mathbb{Q})$ and

$$\text{End}_{\Gamma}(JC) \otimes \mathbb{Q} = (\text{End}(JH) \otimes \mathbb{Q}) \times (\text{End}_{\Gamma}(JF) \otimes \mathbb{Q})^3 = M_2(\mathbb{Q}^{\sqrt{-2}}) \times \mathbb{Q}(i)^3,$$

hence the result. \hfill $\Box$

**Corollary 2** ([ST10]). — Let $\Sigma \subset \mathbb{P}^6$ be the surface of cuboids, defined by

$$t^2 = x^2 + y^2 + z^2, \quad u^2 = y^2 + z^2, \quad v^2 = x^2 + z^2, \quad w^2 = x^2 + y^2.$$  

$\Sigma$ has 48 ordinary nodes; its minimal desingularization $S$ is $\rho$-maximal.

Indeed $\Sigma$ is a quotient of $(C \times C)/\Gamma$ [Bea13]. \hfill $\Box$

(The result has been obtained first in [ST10] with a very different method.)
6. Other examples

6.1. Elliptic modular surfaces. — Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(\mathbb{Z})$ such that $-I \notin \Gamma$. The group $\text{SL}_2(\mathbb{Z})$ acts on the Poincaré upper half-plane $\mathbb{H}$; let $\Delta_\Gamma$ be the compactification of the Riemann surface $\mathbb{H}/\Gamma$. The universal elliptic curve over $\mathbb{H}$ descends to $\mathbb{H}/\Gamma$, and extends to a smooth projective surface $B_\Gamma$ over $\Delta_\Gamma$, the elliptic modular surface attached to $\Gamma$. In [Shi69] Shioda proves that $B_\Gamma$ is $\rho$-maximal.\(^{(1)}\)

Now take $\Gamma = \Gamma(5)$, the kernel of the reduction map $\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/5)$. In [Liv81] Livné constructed a $\mathbb{Z}/5$-covering $X \to B_{\Gamma(5)}$, branched along the sum of the 25 5-torsion sections of $B_{\Gamma(5)}$. The surface $X$ satisfies $c_1^2 = 3c_2 = 225$, hence it is a ball quotient and therefore rigid. By analyzing the action of $\mathbb{Z}/5$ on $H^{1,1}(X)$ Livné shows that $H^{1,1}(X)$ is not defined over $\mathbb{Q}$, hence $X$ is not $\rho$-maximal. This seems to be the only known example of a surface which cannot be deformed to a $\rho$-maximal surface.

6.2. Surfaces with $p_g = K^2 = 1$. — The minimal surfaces with $p_g = K^2 = 1$ have been studied by Catanese [Cat79] and Todorov [Tod80]. Their canonical model is a complete intersection of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,2,2,3,3)$. The moduli space $\mathcal{M}$ is smooth of dimension 18.

Proposition 9. — The $\rho$-maximal surfaces are dense in $\mathcal{M}$.

Proof. — We can replace $\mathcal{M}$ by the Zariski open subset $\mathcal{M}_a$ parametrizing surfaces with ample canonical bundle. Let $S \in \mathcal{M}_a$, and let $f : S \to (B, o)$ be a local versal deformation of $S$, so that $S \cong S_o$. Let $L$ be the lattice $H^2(S, \mathbb{Z})$, and $k \in L$ the class of $K_S$. We may assume that $B$ is simply connected and fix an isomorphism of local systems $R^2f_*\mathbb{Z} \congto L_B$, compatible with the cup-product and mapping the canonical class $[K_{S/B}]$ onto $k$. This induces for each $b \in B$ an isometry $\varphi_b : H^2(S_b, \mathbb{C}) \congto L_C$, which maps $H^{2,0}(S_b)$ onto a line in $L_C$; the corresponding point $\varphi(b)$ of $\mathbb{P}(L_C)$ is the period of $S_b$. It belongs to the complex manifold

$$\Omega := \{ [x] \in \mathbb{P}(L_C) \mid x^2 = 0, \ x \cdot k = 0, \ x \cdot \bar{x} > 0 \}.$$ 

Associating to $x \in \Omega$ the real 2-plane $P_x := \langle \text{Re}(x), \text{Im}(x) \rangle \subset L_R$ defines an isomorphism of $\Omega$ onto the Grassmannian of positive oriented 2-planes in $L_R$.

The key point is that the image of the period map $\varphi : B \to \Omega$ is open [Cat79]. Thus we can find $b$ arbitrarily close to $o$ such that the 2-plane $P_b$ is defined over $\mathbb{Q}$, hence $H^{2,0}(S_b) \oplus H^{0,2}(S_b) = P_b \oplus_\mathbb{R} \mathbb{C}$ is defined over $\mathbb{Q}$. \hfill \Box

Remark 5. — The proof applies to all surfaces with $p_g = 1$ for which the image of the period map is open (for instance to K3 surfaces); unfortunately this seems to be a rather exceptional situation.

---

\(^{(1)}\)I am indebted to I. Dolgachev and B. Totaro for pointing out this reference.
6.3. Todorov surfaces. — In [Tod81] Todorov constructed a series of regular surfaces with $p_g = 1$, $2 \leq K^2 \leq 8$, which provide counter-examples to the Torelli theorem. The construction is as follows: let $K \subset \mathbb{P}^3$ be a Kummer surface. We choose $k$ double points of $K$ in general position (this can be done with $0 \leq k \leq 6$), and a general quadric $Q \subset \mathbb{P}^3$ passing through these $k$ points. The Todorov surface $S$ is the double covering of $K$ branched along $K \cap Q$ and the remaining $16 - k$ double points. It is a minimal surface of general type with $p_g = 1$, $K^2 = 8 - k$, $q = 0$. If moreover we choose $K \rho$-maximal (that is, $K = E^2/\{\pm1\}$, where $E$ is an elliptic curve with complex multiplication), then $S$ is $\rho$-maximal by Proposition 2(b).

Note that by varying the quadric $Q$ we get a continuous, non-constant family of $\rho$-maximal surfaces.

6.4. Double covers. — In [Per82] Persson constructs $\rho$-maximal double covers of certain rational surfaces by allowing the branch curve to acquire some simple singularities (see also [BE87]). He applies this method to find $\rho$-maximal surfaces in the following families:

- Horikawa surfaces, that is, surfaces on the “Noether line” $K^2 = 2p_g - 4$, for $p_g \not\equiv -1 \pmod{6}$;
- Regular elliptic surfaces;
- Double coverings of $\mathbb{P}^2$.

In the latter case the double plane admits (many) rational singularities; it is unknown whether there exists a $\rho$-maximal surface $S$ which is a double covering of $\mathbb{P}^2$ branched along a smooth curve of even degree $\geq 8$.

6.5. Hypersurfaces and complete intersections. — Probably the most natural families to look at are smooth surfaces in $\mathbb{P}^3$, or more generally complete intersections. Here we may ask for a smooth surface $S$, or for the minimal resolution of a surface with rational double points (or even any surface deformation equivalent to a complete intersection of given type). Here are the examples that we know of:

- The quintic surface $x^3yz + y^3zt + z^3tx + t^3xy = 0$ has four $A_9$ singularities; its minimal resolution is $\rho$-maximal [Sch11]. It is not yet known whether there exists a smooth $\rho$-maximal quintic surface.
- The Fermat sextic is $\rho$-maximal (§4, Corollary 1).
- The complete intersection $y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0$ of type $(2, 2, 3)$ in $\mathbb{P}^5$ has 12 nodes; its minimal desingularization is $\rho$-maximal (end of §4).
- The surface of cuboids is a complete intersection of type $(2, 2, 2, 2)$ in $\mathbb{P}^6$ with 48 nodes; its minimal desingularization is $\rho$-maximal (§5, Corollary 2).

7. The complex torus associated to a $\rho$-maximal variety

For a $\rho$-maximal variety $X$, let $T_X$ be the $\mathbb{Z}$-module $H^2(X, \mathbb{Z})/\text{NS}(X)$. We have a decomposition

$$T_X \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$$
defining a weight 1 Hodge structure on $T_X$, hence a complex torus $T := H^{0,2}/p_2(T_X)$, where $p_2 : T_X \otimes \mathbb{C} \to H^{2,0}$ is the second projection. Via the isomorphism $H^{0,2} = H^2(X, \mathcal{O}_X)$, $T_X$ is identified with the cokernel of the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$.

The exponential exact sequence gives rise to an exact sequence

$$0 \to \text{NS}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \to 0,$$

hence to a short exact sequence

$$0 \to T_X \to H^2(X, \mathcal{O}_X^*) \xrightarrow{\partial} H^3(X, \mathbb{Z}),$$

so that $T_X$ appears as the “continuous part” of the group $H^2(X, \mathcal{O}_X^*)$.

**Example 5.** — Consider the elliptic modular surface $B_\Gamma$ of Section 6.1. The space $H^0(B_\Gamma, K_{B_\Gamma})$ can be identified with the space of cusp forms of weight 3 for $\Gamma$; then the torus $T_{B_\Gamma}$ is the complex torus associated to this space by Shimura (see [Shi69]).

**Example 6.** — Let $X = C \times C'$, with $JC$ isogenous to $E^g$ and $JC'$ to $E^{g'}$ (Proposition 5). The torus $T_X$ is the cokernel of the map

$$i \otimes i' : H^1(C, \mathbb{Z}) \otimes H^1(C', \mathbb{Z}) \to H^1(C, \mathcal{O}_C) \otimes H^1(C', \mathcal{O}_{C'}),$$

where $i$ and $i'$ are the embeddings

$$H^1(C, \mathbb{Z}) \hookrightarrow H^1(C, \mathcal{O}_C) \quad \text{and} \quad H^1(C', \mathbb{Z}) \hookrightarrow H^1(C', \mathcal{O}_{C'}).$$

We want to compute $T_X$ up to isogeny, so we may replace the left hand side by a finite index sublattice. Thus, writing $E = \mathbb{C}/\Gamma$, we may identify $i$ with the diagonal embedding $\Gamma^g \hookrightarrow \mathbb{C}^g$, and similarly for $i'$; therefore $i \otimes i'$ is the diagonal embedding of $(\Gamma \otimes \Gamma)^{g,g'}$ in $\mathbb{C}^{g,g'}$. Put $\Gamma = \mathbb{Z} + \tau \mathbb{Z}$; the image $\Gamma'$ of $\Gamma \otimes \Gamma$ in $\mathbb{C}$ is spanned by $1, \tau, \tau^2$; since $E$ has complex multiplication, $\tau$ is a quadratic number, hence $\Gamma$ has finite index in $\Gamma'$. Finally we obtain that $T_X$ is isogenous to $E^{g,g'}$.

For the surface $X = (C \times C)/\Gamma$ studied in §5 an analogous argument shows that $T_X$ is isogenous to $A = E^4_1 \times E^3_{\sqrt{-2}}$. This is still an abelian variety of type CM, in the sense that $\text{End}(A) \otimes \mathbb{Q}$ contains an étale $\mathbb{Q}$-algebra of maximal dimension $2\dim(A)$. There seems to be no reason why this should hold in general. However it is true in the special case $h^{2,0} = 1$ (e.g. for holomorphic symplectic manifolds):

**Proposition 10.** — If $h^{2,0}(X) = 1$, the torus $T_X$ is an elliptic curve with complex multiplication.

**Proof.** — Let $T_X'$ be the pull back of $H^{2,0} + H^{0,2}$ in $H^2(X, \mathbb{Z})$; then $p_2(T_X')$ is a sublattice of finite index in $p_2(T_X)$. Choosing an ample class $h \in H^2(X, \mathbb{Z})$ defines a quadratic form on $H^2(X, \mathbb{Z})$ which is positive definite on $T_X$. Replacing again $T_X'$ by a finite index sublattice we may assume that it admits an orthogonal basis $(e, f)$ with $e^2 = a$, $f^2 = b$. Then $H^{2,0}$ and $H^{0,2}$ are the two isotropic lines of $T_X' \otimes \mathbb{C}$; they are spanned by the vectors $\omega = e + \tau f$ and $\overline{\omega} = e - \tau f$, with $\tau^2 = -a/b$. We have $e = \frac{1}{2}(\omega + \overline{\omega})$ and $f = \frac{1}{2\tau}(\omega - \overline{\omega})$; therefore multiplication by $\frac{1}{2\tau} \overline{\omega}$ induces an
isomorphism of $\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ onto $H^{0,2}/\mathbb{Z}_2(T_X^*)$, hence $T_X$ is isogenous to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ and

$$\text{End}(T_X) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-\text{disc}(T_X^*)}).$$

### 8. Higher codimension cycles

A natural generalization of the question considered here is to look for varieties $X$ for which the group $H^{2p}(X,\mathbb{Z})_{\text{alg}}$ of algebraic classes in $H^{2p}(X,\mathbb{Z})$ has maximal rank $h^{p,p}$.

**Proposition 11.** Let $F^n_d$ be the Fermat hypersurface of degree $d$ and even dimension $n = 2\nu$. For $d = 3,4$, the group $H^n(F^n_d,\mathbb{Z})_{\text{alg}}$ has maximal rank $h^{\nu,\nu}$.

**Proof.** According to [Shi79] we have

$$\text{rk} H^n(F^n_d,\mathbb{Z})_{\text{alg}} = 1 + \frac{n!}{(\nu)!^2} \quad \text{and} \quad \text{rk} H^n(F^n_d,\mathbb{Z})_{\text{alg}} = \sum_{k=0}^{\nu+1} \frac{(n+2)!}{(k)!^2(n+2-2k)!}.$$

On the other hand, let $R^n_d := \mathbb{C}[X_0,\ldots,X_{n+1}]/(X_0^{d-1},\ldots,X_{n+1}^{d-1})$ be the Jacobian ring of $F^n_d$; Griffiths theory [Gri69] provides an isomorphism of the primitive cohomology $H^n_{\text{pr}}(F^n_d)_\mathbb{C}$, with the component of degree $(\nu+1)(d-2)$ of $R^n_d$. Since this ring is the tensor product of $(n+2)$ copies of $\mathbb{C}[T]/(T^{d-1})$, its Poincaré series $\sum_k \dim(R^n_d)_k T^k$ is $(1 + T + \cdots + T^{d-2})^{n+2}$. Then an elementary computation gives the result. □

In the particular case of cubic fourfolds we have more examples:

**Proposition 12.** Let $F$ be a cubic form in 3 variables, such that the curve $F(x,y,z) = 0$ in $\mathbb{P}^2$ is an elliptic curve with complex multiplication; let $X$ be the cubic fourfold defined by $F(x,y,z) + F(u,v,w) = 0$ in $\mathbb{P}^5$. The group $H^3(X,\mathbb{Z})_{\text{alg}}$ has maximal rank $h^{3,2}(X)$.

**Proof.** Let $u$ be the automorphism of $X$ defined by

$$u(x,y,z;u,v,w) = (x,y,z;\omega u,\omega v,\omega w).$$

We observe that $u$ acts trivially on the (one-dimensional) space $H^3(X)$. Indeed Griffiths theory [Gri69] provides a canonical isomorphism

$$\text{Res} : H^3(\mathbb{P}^5, \mathbb{Q}_{\text{alg}}(2X)) \overset{\sim}{\longrightarrow} H^3_{\text{alg}}(X);$$

the space $H^3(\mathbb{P}^5, \mathbb{Q}_{\text{alg}}(2X))$ is generated by the meromorphic form $\Omega/G^2$, with

$$\Omega = xdy \wedge dz \wedge du \wedge dw - ydx \wedge dz \wedge du \wedge dv \wedge dw + \cdots,$$

$$G = F(x,y,z) + F(u,v,w).$$

The automorphism $u$ acts trivially on this form, and therefore on $H^3(X)$.

Let $F$ be the variety of lines contained in $X$. We recall from [BD85] that $F$ is a holomorphic symplectic fourfold, and that there is a natural isomorphism of Hodge structures $\alpha : H^3(X,\mathbb{Z}) \overset{\sim}{\longrightarrow} H^2(F,\mathbb{Z})$. Therefore the automorphism $u_F$ of $F$ induced by $u$ is symplectic. Let us describe its fixed locus.
The fixed locus of $u$ in $X$ is the union of the plane cubics $E$ given by $x = y = z = 0$ and $E'$ given by $u = v = w = 0$. A line in $X$ preserved by $u$ must have (at least) two fixed points, hence must meet both $E$ and $E'$; conversely, any line joining a point of $E$ to a point of $E'$ is contained in $X$, and preserved by $u$. This identifies the fixed locus $A$ of $u$ to the abelian surface $E \times E'$. Since $u$ is symplectic $A$ is a symplectic submanifold, that is, the restriction map $H^{2,0}(F) \rightarrow H^{2,0}(A)$ is an isomorphism. By our hypothesis $A$ is $\rho$-maximal, so $F$ is $\rho$-maximal by Proposition 2. Since $\alpha$ maps $H^4(X, \mathbb{Z})_{alg}$ onto $\text{NS}(F)$ this implies the Proposition. □

References


Manuscript received January 2, 2014
accepted May 16, 2014

Arnaud Beauville, Laboratoire J.-A. Dieudonné, UMR 7351 du CNRS, Université de Nice
Parc Valrose, F-06108 Nice cedex 2, France
E-mail : arnaud.beauville@unice.fr
Url : http://math1.unice.fr/~beauville/